All ASD complex and real 4-dimensional Einstein spaces with $\Lambda \neq 0$ admitting a nonnull Killing vector

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Motivations and goals

- In heavenly spaces with $\Lambda = 0$ and in hyperheavenly spaces ten Killing equations were reduced to single *master equation*. Such reduction in heavenly spaces with $\Lambda \neq 0$ was unknown.
- What is the simplest form of the metric and the form of the reduced heavenly equation for the complex ASD spaces with Λ admitting a nonnull Killing vector?
- How to obtain all real slices of this complex metric?

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Self-dual null strings

- 2-dimensional holomorphic distribution $\mathcal{D} = \{\mu_A \nu_{\dot{M}}, \mu_A \rho_{\dot{M}}\}$, $\nu_{\dot{M}} \rho^{\dot{M}} \neq 0$ is integrable in the Frobenius sense if spinor μ_A satisfies the equations
$$\mu^B \mu^C \nabla_{B\dot{A}} \mu_C = 0$$
- Integral manifolds of distribution $\mathcal{D}$ are called *self-dual null strings (SD null strings)* and they constitute the *congruence of self-dual null strings*.
- There are two essentially different types of the congruences of the null strings: expanding and nonexpanding. Nonexpanding case corresponds to the null strings which are parallely propagated.

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Self-dual null strings

- In Einstein spaces the existence of the SD null strings implies that the SD Weyl tensor C_{ABCD} is algebraically degenerated and spinor μ_A is the undotted multiple Penrose spinor.

$$C_{ABCD}\mu^A\mu^B\mu^C = 0$$

- The number of independent congruences of the SD null strings is equal to the number of multiple undotted Penrose spinors.
- There are infinitely many independent congruences of SD null strings in the ASD Einstein spaces. Moreover, if $\Lambda \neq 0$, all of them are expanding.

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Definitions of heavenly spaces

Definition (Heavenly space)

$\mathcal{H}$-space with cosmological constant Λ is a 4 - dimensional complex analytic differential manifold endowed with a holomorphic Riemannian metric ds^2 satisfying the vacuum Einstein equations with Λ, $R_{ab} = -\Lambda g_{ab}$, $\Lambda \neq 0$, and such that the SD or ASD part of the Weyl tensor vanishes.

In what follows we assume that $C_{ABCD} = 0$ so we deal with ASD Einstein spaces with Λ (left-flat heavenly spaces).

Heavenly spaces in Plebański - Robinson coordinates

- The metric of the ASD, Einstein space with Λ can be locally brought to the form

$$ds^2 = \phi^{-2}\Big\{2\tau^{-1}(d\eta dw - d\phi dt) + 2\left(-\phi W_{\eta\eta} + \frac{\Lambda}{6\tau^2}\right)dt^2 + 4\,(W_\eta - \phi W_{\eta\phi})\,dwdt + 2\,(2\,W_\phi - \phi W_{\phi\phi})\,dw^2\Big\}$$

where (ϕ, η, w, t) are *Plebański - Robinson coordinates* and the complex constant parameter τ can be chosen as convenient. $W = W(\phi, \eta, w, t)$ is *the key function*.

- Einstein equations can be reduced to *heavenly equation with* Λ

$$W_{\eta\eta}W_{\phi\phi} - W_{\eta\phi}W_{\eta\phi} + 2\phi^{-1}W_\eta W_{\eta\phi} - 2\phi^{-1}W_\phi W_{\eta\eta} + (\tau\phi)^{-1}(W_{w\eta} - W_{t\phi}) - \frac{\Lambda}{6\tau^2}\phi^{-1}W_{\phi\phi} = 0$$

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Killing equations

- In ASD Einstein spaces with $\Lambda \neq 0$ and $C_{\dot{A}\dot{B}\dot{C}\dot{D}} \neq 0$ proper homothetic and proper conformal Killing symmetries are not allowed.
- Spinorial Killing equations read

$$\nabla_{(A}^{\ \ (\dot{B}} K_{C)}^{\ \ \dot{D})} = 0 \ , \quad \nabla^{N\dot{N}} K_{N\dot{N}} = 0$$

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Invariant which characterize Killing vector

- Any Killing vector can be characterized by the invariant $l := l_{AB} l^{AB}$ where spinor l_{AB} is defined by the relation

$$l_{AB} := \frac{1}{2} \nabla_{(A}{}^{\dot{N}} K_{B)\dot{N}}$$

- It can be proved, that in ASD Einstein spaces with $\Lambda \neq 0$

$$l = 0 \Longleftrightarrow \text{Killing vector is null}$$
$$l \neq 0 \Longleftrightarrow \text{Killing vector is nonnull}$$

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The master equation

Ten Killing equations in ASD Einstein spaces with Λ can be reduced to one equation called *the master equation*. Unfortunately, this equation is much more complicated then the similar equation in hyperheavenly spaces.

Why?

What is the best way to use the heavenly structure?

- There are infinitely many congruences of SD null strings. Which one should we choose?
- Decomposing spinor l_{AB} according to the formula $l_{AB} = m_{(A}n_{B)}$ one finds, that both m_A and n_A satisfy the null string equations.
- Then the best choice is to use the congruence of the null strings generated by the spinor m_A. The complex spinor transformation allows to set $l_{11} = 0$.

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Metric in Plebański - Robinson coordinates

- Any nonnull Killing vector can be brought to the form $K = \partial_t$
- From the master equation it follows, that the key function becomes the function of three variables $W = W(\phi, \eta, w)$.

The metric takes the form

$$ds^2 = \phi^{-2} \Big\{ 2\tau^{-1}(d\eta dw - d\phi dt) + 2\left(-\phi W_{\eta\eta} + \frac{\Lambda}{6\tau^2}\right) dt^2 + 4\,(W_\eta - \phi W_{\eta\phi})\, dwdt + 2\,(2\,W_\phi - \phi W_{\phi\phi})\, dw^2 \Big\}$$

Heavenly equation with Λ reduces to the equation

$$W_{\eta\eta}W_{\phi\phi} - W_{\eta\phi}W_{\eta\phi} + 2\phi^{-1}W_\eta W_{\eta\phi} - 2\phi^{-1}W_\phi W_{\eta\eta} + (\tau\phi)^{-1}W_{w\eta} - \frac{\Lambda}{6\tau^2}\phi^{-1}W_{\phi\phi} = 0$$

Metric in LeBrun form (LeBrun, 1991)

In new coordinate system (X, Y, Z, T) the Killing vector has the form $K = \partial_Z$ and the metric can be locally brought to the form

$$ds^2 = \frac{V}{T^2}\left(e^U(dX^2 - dY^2) + dT^2\right) - \frac{1}{VT^2}(dZ + \alpha)^2$$

where

$$V := \frac{3}{2}\frac{TU_T - 2}{\Lambda}$$

and the 1-form α fulfills the equation

$$-\frac{2}{3}\Lambda\, d\alpha = (e^U)_T\, dX \wedge dY - TdX \wedge dU_Y + TdU_X \wedge dY$$

Heavenly equation with Λ can be brought to the Boyer - Finley - Plebański equation for $U = U(T, X, Y)$

$$(e^U)_{TT} + U_{XX} - U_{YY} = 0$$

Invariant, which characterizes the Killing vector reads $l_{AB}l^{AB} = -\frac{2}{9}\frac{\Lambda^2}{T^2} \neq 0.$

Metric in the Σ-formalism

Changing the coordinates $(T, X, Y, Z) \to (\varrho, \varphi, \xi, \upsilon)$ according to the formulas

$$Z = -\varrho \ , \quad T = \varphi \ , \quad Y = \xi - \upsilon \ , \quad X = -\xi - \upsilon$$

and using the potential Σ defined by the relation

$$U =: \ln \Sigma_\xi$$

we arrive to the alternate form of the metric:

Metric in the Σ-formalism

In coordinates (φ, z, ϱ, v) the Killing vector is $K = \partial_\varrho$ and the metric can be locally brought to the form

$$ds^2 = \varphi^{-2}\Big\{ -\frac{2}{\tau}\, d\varphi d\varrho + \frac{2\Lambda}{3\tau^2}\frac{\Sigma_\xi}{\Omega_\xi}\, d\varrho^2 + \frac{4}{\tau}\frac{\Sigma_\xi \Omega_\varphi}{\Omega_\xi}\, dv d\varrho + \frac{6}{\Lambda}\frac{\Sigma_\xi \Omega_\varphi^2}{\Omega_\xi}\, dv^2 - \frac{6}{\Lambda}\Omega_\varphi\, d\varphi dv - \frac{6}{\Lambda}\Omega_\xi\, d\xi dv \Big\}$$

where

$$\Omega := 2\Sigma - \varphi\Sigma_\varphi \ , \quad \Omega_\xi \neq 0$$

Heavenly equation with Λ takes the form

$$\Sigma_{\xi v} + \Sigma_\xi \Sigma_{\varphi\varphi} = 0$$

Real neutral slices

- In neutral signature the spinor l_{AB} is real

$$l_{AB} = \bar{l}_{AB}$$

- Using decomposition $l_{AB} = m_{(A}n_{B)}$ we obtain two solutions:

$$1)\quad m_A \text{ and } n_A \text{ are real} \quad \Longrightarrow \quad l_{AB}l^{AB} < 0$$
$$2)\quad m_A \text{ and } n_A \text{ are complex: } m_A = \pm\bar{n}_A \quad \Longrightarrow \quad l_{AB}l^{AB} > 0$$

- The first case can be obtained from the complex metric in LeBrun form by direct real slice
- There are no real null strings related to the Killing vector in the second case
- Solution: to perform the complex transformation of the coordinates, namely $T \to iT$, $Y \to iY$ and then take the real slice of obtained complex metric

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Real neutral slices (Högner, 2012)

For the real ASD Einstein spaces in neutral $(++--)$ signature with $\Lambda \neq 0$ admitting nonnull Killing vector $K = \partial_z$ we obtained

$$ds^2 = \frac{V}{t^2}\left(e^U(dx^2 \pm dy^2) \mp dt^2\right) - \frac{1}{Vt^2}(dz+\alpha)^2$$

where

$$V := \pm\frac{tU_t - 2}{2\Lambda}$$

and the 1-form α satisfies

$$2\Lambda\, d\alpha = (e^U)_t\, dx \wedge dy - t\, dx \wedge dU_y \mp t\, dU_x \wedge dy$$

The function U satisfies BFP equation

$$(e^U)_{tt} \mp U_{xx} - U_{yy} = 0$$

Moreover

- case $l_{AB}l^{AB} > 0$ corresponds to the upper signs
- case $l_{AB}l^{AB} < 0$ corresponds to the lower signs

Real Euclidean slices

- In Euclidean signature $(++++)$ there are no congruences of null strings
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Real Euclidean slices (Przanowski, 1991; Tod, 2006)

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$$ds^2 = \frac{V}{t^2}\left(e^U(dx^2 + dy^2) + dt^2\right) + \frac{1}{Vt^2}(dz + \alpha)^2$$

where

$$V = \frac{tU_t - 2}{4\Lambda}$$

and the 1-form α satisfies

$$-4\Lambda\, d\alpha = (e^U)_t\, dx \wedge dy + t\, dx \wedge dU_y + t\, dU_x \wedge dy$$

The function U satisfies BFP equation

$$(e^U)_{tt} + U_{xx} + U_{yy} = 0$$

Real Lorentzian slices

- Lorentzian slices exist only if $\bar{C}_{ABCD} = C_{\dot{A}\dot{B}\dot{C}\dot{D}}$
- The only Lorentzian slices which can be obtained from the considered metrics after setting $C_{\dot{A}\dot{B}\dot{C}\dot{D}} = 0$ are de-Sitter metrics.
- To obtain complex de-Sitter metric without any loss of generality we put

$$W = 0 \quad \text{in Plebański - Robinson coordinates}$$
$$U = 0 \quad \text{in Le Brun coordinates}$$
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Conclusions

- Hyperheavenly and heavenly spaces formalism in Plebański - Robinson coordinates seems to be "designed" for real problems in neutral $(++--)$ signature. However, not all real neutral classes can be obtained from generic complex metric by direct real slice technique. In some cases additional complex transformation of the coordinates is needed.
- Similar technique allowed to obtain real Lorentzian metric admitting null Killing vector of the type $[\mathrm{II}]$ from the complex hyperheavenly metric of the type $[\mathrm{II}] \otimes [\mathrm{II}]$. How to use this technique in other cases? Are there any chances to obtain new vacuum solutions of Einstein field equations in Lorentzian signature from hyperheavenly metrics (Plebański programme)?
- Problem: to find all solutions of BFP equation which give conformally flat spaces (Le Brun form or Σ-formalism?)

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