Self-dual conformal gravity

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- MD, Paul Tod `arXiv:1304.7772.`, Comm. Math. Phys. (2014).

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Conformal to Einstein

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- Curvature decomposition: Riemann=Weyl+Ricci+Scalar

$$[\nabla_a, \nabla_b]V^c = R_{ab}{}^c{}_d V^d$$

$$R_{abcd} = C_{abcd} + \frac{2}{n-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) - \frac{2}{(n-1)(n-2)} R g_{a[c}g_{d]b}.$$

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- (M, g) is
  1. Einstein, if $R_{ab} = \frac{1}{n} R g_{ab}$.
  2. Ricci–flat if $R_{ab} = 0$.
  3. Conformal to Einstein (Ricci–flat) if there exists $\Omega : M \to \mathbb{R}^+$ such that $\hat{g} = \Omega^2 g$ is Einstein (Ricci–flat).

- (M, g) Lorentzian four-manifold. Does there exist $\Omega : M \longrightarrow \mathbb{R}^+$, and a local coordinate system (x, y, z, t) such that

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  2. $n = 3$. Einstein=constant curvature. Need $Y_{abc} = 0$ (Cotton tensor).
  3. $n = 4$?? Brinkman 1920s, Szekeres 1963, Kozameh–Newman–Tod 1985.

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- Conformal gravity:

$$\int_M |C|^2 \text{vol}_g \longrightarrow \text{Euler-Lagrange} \longrightarrow B_{ab} = 0.$$

ANTI-SELF-DUALITY (GENERICITY FAILS)

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- Signature $(4, 0)$ or $(2, 2)$. (Lorentzian+ASD=conformal flatness).
- Question: Given a 4-manifold (M, g) with ASD Weyl tensor, how to determine whether g is conformal to a Ricci–flat metric?

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- Conformally invariant if

$$g \to \Omega^2 g, \quad \varepsilon \to \Omega\, \varepsilon, \quad \varepsilon' \to \Omega\, \varepsilon', \quad \pi \to \Omega\, \pi.$$

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$$\nabla\pi - \alpha \otimes \epsilon' = 0, \quad \nabla\alpha + \pi \lrcorner P = 0,$$

where $\alpha \in \Gamma(\mathbb{S})$, and $P_{ab} = (1/12)Rg_{ab} - (1/2)R_{ab}$.

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- Spinor conjugation in Riemannian signature: $\sigma : \mathbb{S}' \to \mathbb{S}'$.

$$\pi = (p, q), \quad \sigma(\pi) = (-\bar{q}, \bar{p}), \quad \text{so} \quad \pi \in \text{Ker } \mathbb{D} \leftrightarrow \sigma(\pi) \in \text{Ker } \mathbb{D}.$$

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- Theorem 1. There is a one-to-one correspondence between parallel sections of $(E, \mathcal{D})$ and Ricci–flat metrics in an ASD conformal class.

Tensor Obstructions

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- Necessary conditions: rank $(\mathcal{R}) = 2$. Sufficient conditions: rank $(\mathcal{R},\ \mathcal{D}\mathcal{R}) = 2$.
- Set $V_a = 4|C|^{-2}C^{bcd}{}_a\nabla^e C_{bcde}$.
  Theorem 2. An ASD Riemannian metric g is conformal to a Ricci–flat metric if and only if

$$\begin{aligned} \text{det}(\mathcal{R}) &:= 4\nabla^e C_{bcde}\nabla_f C^{bcdf} - |V|^2|C|^2 = 0, \quad \text{and} \\ T_{ab} &:= P_{ab} + \nabla_a V_b + V_a V_b - \frac{1}{2}|V|^2 g_{ab} = 0. \end{aligned}$$

Example

- Two parameter family of Riemannian metrics (LeBrun 1988)

$$g = f^{-1}dr^2 + \frac{1}{4}r^2(\sigma_1^2 + \sigma_2^2 + f\sigma_3^2), \quad \text{where} \quad f = 1 + \frac{A}{r^2} + \frac{B}{r^4},$$

where σ_j are 1-forms on $SU(2)$ such that $d\sigma_1 = \sigma_2 \wedge \sigma_3$ etc.

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- Kahler and vanishing Ricci scalar, so ASD. Conformal to Ricci flat?

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- g is ASD Kahler, $\Omega^2 g$ is hyper-Kahler! Are there more such examples?? (Clue: The Kahler form for g becomes a conformal Killing–Yano tensor for $\hat{g}$).

Thank You!