# Goldberg-Sachs theorem: extensions

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*Reference GS thm:* a conformally Einstein conformal complex fourfold has algebraically special SD Weyl curvature if and only if it admits an integrable distribution of SD null planes.

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Classical twistor fibration (complexified):

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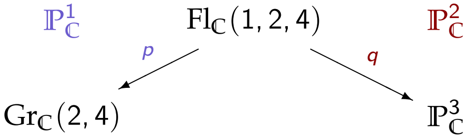
**Kerr's theorem:**

$$(\text{surfaces in } \mathbb{P}^3_{\mathbb{C}}) \simeq (\textit{integrable} \text{ SD null rk 2 distributions})$$

# Kerr

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$$\mathscr{N} = dp|_{q^{-1}X} \ker dq \qquad \Longrightarrow \qquad \text{integrable}$$

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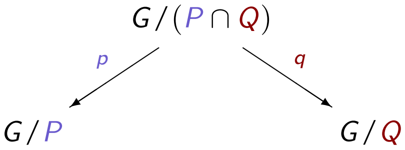

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Lift $\mathscr{D} \subset T\mathscr{C}$ splits naturally: $\mathscr{D} = \ker dp \oplus \mathscr{A}$.

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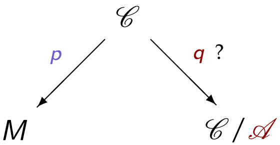

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Identify:

- $\mathbb{P}S_+ \simeq \mathscr{C}$ projectivisation of half-spinor bundle,
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The obstruction is precisely $\Psi_+$.

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- Classically: $C = \mathbb{P}^1_{\mathbb{C}}$ and $V = \mathscr{O}(4)$.

# Generalisation

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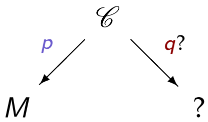


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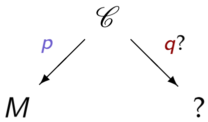


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Classically $\star$ is 'being conformally Einstein' (actually less),
† is integrability of the corresponding SD rk 2 null distribution.

**Nurowski:** classical † $\iff$ being parallel with respect to *some* Weyl connection.

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## Theorem

*Let $M$ be a $2n$-dimensional complex conformal manifold, $\mathscr{C} \to M$ the bundle of SD null $n$-planes, and $\zeta : \mathscr{C} \to M$ a section parallel with respect to some Weyl connection.*

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This theorem: additionally assuming existence of an *adapted Weyl connection* gives a *sufficient* (but not necessary) condition.

# Setup

$G = \mathrm{Spin}_{\mathbb{C}}(2n+2)$, $P = \mathrm{Stab}\,\ell$, $Q = \mathrm{Stab}\,N$
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$(P, Q)$-bigrading:

$$\mathfrak{g}_{-1} = \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,0}, \quad \mathfrak{g}_0 = \mathfrak{g}_{0,-1} \oplus \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{0,1}, \quad \mathfrak{g}_1 = \mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,1}.$$

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This extends to representations; in particular the space of algebraic Weyl tensors has components in degrees $(0,-1), (0,0), \ldots, (0,4)$.

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- Weyl connections preserving $\zeta$ as torsion-free principal connections on $\mathscr{E}_\zeta$.

# Consequences

Fix a Weyl connection preserving $\zeta : M \to \mathscr{C}$. Its curvature

$$R : \mathscr{E}_\sigma \to \Lambda^2 \mathfrak{g}_1 \otimes \mathfrak{g}_0$$

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*Let $R \in \Lambda^2 \mathfrak{g}_1 \otimes \mathfrak{g}_0$ be an algebraic curvature tensor, with Weyl and Schouten components $\Psi$ and $\mathsf{P} \in \mathfrak{g}_1 \otimes \mathfrak{g}_1$. Suppose $R$ projects trivially onto $\Lambda^2 \mathfrak{g}_1 \otimes \mathfrak{g}_{0,-1}$. Then:*

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2. $\mathrm{P}$ *is skew mod $Q$-degree* $1$,
3. $\Psi = -\mathrm{P}$ *mod $Q$-degree* $1$

*for suitable inclusion of $\Lambda^2 \mathfrak{g}_{1,0}$ into the $Q$-degree $0$ subspace of algebraic Weyl tensors.*

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Transforming to an arbitrary Weyl connection $\hat{\nabla}$:

$$\Psi \otimes \Psi - \mathrm{Alt}\,\hat{\mathrm{P}} \otimes \Psi + \lambda\hat{A} = 0 \quad \text{mod } Q\text{-degree } 1$$

where $\lambda$ is a first-order linear differential operator preserving $Q$-degree.

# Conclusion

Choosing $\hat{\nabla}$ to be a Levi-Civita connection for a metric in the conformal class: $\Psi \otimes \Psi + \lambda \hat{A} = 0$ mod $Q$-degree $1$.

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This proves the Theorem.