The geodesic structure of some spacetimes with symmetries — local and global properties

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Motivation: 'twin paradox'

The 'twin paradox' in SR has 3 levels of comprehending:
— why there is at all the asymmetry between the twins,
— why the accelerated twin is younger ('reverse triangle inequality'),
— what happens in a curved spacetime.
Curvature ⇒ multitude od diverse results. Purely geometrical problem : which timelike curve joining two given points is the longest one? There is no *shortest* timelike line.
The problem deeply enters into the geodesic structure of the spacetime — this is why it is worth studying.
The problem: local and global. Here: only a brief introduction.

Locally maximal timelike curves

A bundle of nearby timelike curves emanating from p and intersecting at p_1, we seek for the longest one. It contains a geodesic γ_0, $\gamma_0(0) = p$, $\gamma_0(s_1) = p_1$.
The bundle may contain other geodesics γ_ε infinitesim. close to γ_0, $|\varepsilon| \ll 1$.
$x^\mu(s)$ — coordinates of γ_0, $u^\mu(s)$ — tangent to γ_0,
$\bar{x}^\mu(s, \varepsilon)$ - coordinates of γ_ε,

$$\bar{x}^\mu(s, \varepsilon) = x^\mu(s) + \varepsilon Z^\mu(s) + \delta^2 x^\mu(s) + \ldots + \delta^n x^\mu(s) + \ldots,$$

$Z^\mu(s)$ — the *connecting vector* in the linear approximation, (the *geodesic deviation vector*, *Jacobi vector field*), $Z^\mu\, u_\mu = 0$,
the $n-$th deviation $\delta^n x^\mu = O(\varepsilon^n)$ for $n > 1$ is not a vector.

$Z^\mu(s)$ satisfies the *geodesic deviation equation* (GDE)

$$\frac{D^2}{ds^2} Z^\mu = R^\mu{}_{\alpha\beta\gamma}\, u^\alpha\, u^\beta\, Z^\gamma$$

(derived in the linear approximation).
If $Z^\mu \neq 0$ and $Z^\mu(0) = Z^\mu(s_0) = 0$ for $0 < s_0 < s_1 \Rightarrow$
— either γ_ε intersects γ_0 at $q = \gamma_0(s_0)$ or
— γ_ε is closer to γ_0 at q than at ε–distance.
In both cases it is interpreted that γ_ε intersects γ_0 at q on the segment pp_1.
$q = \gamma_0(s_0)$ — point *conjugate* to p on γ_0.

Proposition (Hawking & Ellis)

A timelike geodesic γ_0 has the *locally maximal* length from p to p_1 iff there is NO point conjugate to p on the segment pp_1.

Conversely:

if there is a point q conjugate to p on the segment pp_1 of γ_0 then there exists a nearby timelike curve λ (not necessarily geodesic) from p to p_1 which is longer than γ_0, $s(\lambda) > s(\gamma_0)$.

The locally longest curve (geodesic) always exists in globally hyperbolic spacetimes.

CAdS — exist pairs of points connected by timelike lines but none is a geodesic and none is maximal (neither locally nor globally).

Criterion (Hawking & Ellis)
If $R_{\alpha\beta}\, u^{\alpha}\, u^{\beta} \geq 0$ (SEC) on a timelike geodesic γ and if the tidal force $R_{\mu\alpha\nu\beta}\, u^{\alpha}\, u^{\beta} \neq 0$ at some point p_0 on γ_0, then there is a pair of conjugate points p and q on γ_0 (if sufficiently extended).

LOCALLY LONGEST GEODESICS

We seek for maximal segments of geodesics $\Rightarrow$ seek for conjugate points $\Rightarrow$ seek for zeros of Jacobi fields $Z^{\mu}(s)$.

γ_0 — chosen timelike geodesic, u^μ — tangent to γ_0.
Replace D^2/ds^2 in GDE by d^2/ds^2.
Choose tetrad $\{e_A{}^\mu\}$, $A = 0, 1, 2, 3$, orthonormal and parallelly transported along γ_0,

$$e_0{}^\mu \equiv u^\mu, \quad e_A{}^\mu \, e_{B\mu} = \eta_{AB} = \mathrm{diag}(1, -1, -1, -1), \quad \frac{D}{ds} e_A{}^\mu = 0,$$

$\{e_a{}^\mu\}$, $a = 1, 2, 3$ —spacelike triad orthogonal to γ_0.
Expand

$$Z^\mu = \sum_{a=1}^{3} Z_a \, e_a{}^\mu, \qquad \text{then GDE}$$

$$\frac{d^2}{ds^2} Z_a = -e_a^\mu \, R_{\mu\alpha\beta\gamma} \, u^\alpha \, u^\beta \sum_{b=1}^{3} Z_b \, e_b{}^\gamma,$$

Z_a — 3 *Jacobi scalars*.

Killing vector $K^\mu \Rightarrow$ first integral of GDE

$$K_\mu \frac{D}{ds} Z^\mu - Z^\mu \frac{D}{ds} K_\mu = \mathrm{const.}$$

PROCEDURE

— Choose a spacetime with some some symmetries (Killing vectors).

— Choose geometrically interesting timelike geodesic with explicit $x^\alpha = x^\alpha(\tau)$, e. g. $\tau = s$.

— Choose the spacelike triad ${e_a}^\mu$ as above.

— Solve GDE

$$\frac{d^2}{ds^2} Z_a = -e_a^\mu \, R_{\mu\alpha\beta\gamma} \, u^\alpha \, u^\beta \sum_{b=1}^{3} Z_b \, {e_b}^\gamma$$

applying the first integrals and find a *generic* solution $Z_a(\tau)$.

— Consider *all possible special* solutions with $Z_a(\tau = 0) = 0$ and seek for their zeros,

$$Z_a(\tau_0) = 0, \qquad \tau_0 > 0.$$

Then the geodesic $x^\alpha = x^\alpha(\tau)$ is *locally maximal* on the segment $0 \le \tau < \tau_0$.
This is a *fully algorithmic* procedure for checking that the given geodesic is the unique *locally longest* curve between given points.

COMPUTATIONS

— Static spherically symmetric (SSS) spacetimes: de Sitter, CAdS, Schwarzschild, Reissner–Nordström and Bertotti–Robinson,

— ultrastatic SSS spacetimes: Barriola–Vilenkin monopole,

— cosmological Robertson–Walker spacetimes.

Geodesics: radial and circular (if exist).

A variety of diverse results. Spacetimes with similar symetries may have different geodesic structure: dS and CAdS.

One generic result:

all stable circular (if exist) geodesics in all SSS spacetimes contain 3 infinite sequences of conjugate points to any initial point and only one sequence has clear geometric meaning.

Globally maximal timelike curves

The search for globally longest curve from p to p_1 is different conceptually and in practice.
Ω — the space of *all* future directed timelike piecewise smooth curves from p to p_1. Each curve λ has length $s(\lambda) > 0$.
$\lambda \in \Omega$ is *globally maximal* if it is the longest curve in $\Omega \Leftrightarrow$ its length is equal to the *Lorentzian distance function*

$$s(\lambda) = d(p, p_1).$$

The maximal curve is always a geodesic (non unique).
In globally hyperbolic spacetimes for any pair $p \prec\prec p_1$ there is globally maximal geodesic $\gamma \in \Omega$.
In the global problem the conjugate point is replaced by a *cut point*.

A geodesic γ, $0 \leq s < a$, is usually not globally maximal beyond some its segment.
Let

$$s_0 \equiv \sup\{s \in [0, a) : d(\gamma(0), \gamma(s)) = s\},$$

if $s_0 < a \Rightarrow \gamma(s_0)$ — *future timelike cut point* of $\gamma(0)$ on γ.
s_0 — the length of the longest maximal segment of γ.
This means that on the segment:
— from $\gamma(0)$ to all $\gamma(s)$, $s < s_0$ — γ is the unique globally maximal,
— from $\gamma(0)$ to $\gamma(s_0)$ — γ is globally maximal (not unique),
— from $\gamma(0)$ to $\gamma(s_1)$, $s_1 > s_0$ — there are curves longer than γ.
The future cut point of $p = \gamma(0)$ along γ comes *no later* than the first future conjugate point to p. A geodesic may contain only the cut point and no conjugate points.

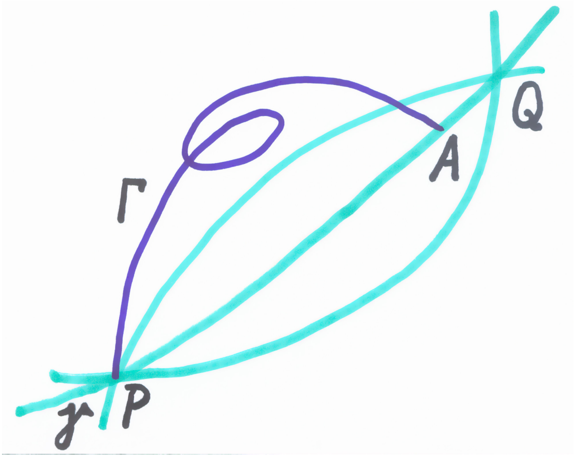

Figure: Q — nearest conjugate point of P on γ, A — the future cut point of P, Γ and γ — two longest timelike geodesics from P to A.

GLOBAL LORENTZIAN GEOMETRY

Our current knowledge:

There is a number of ‘existence theorems’ on maximal timelike geodesics in various spacetimes valid if some global conditions are satisfied.

They are not ‘constructive’: do not indicate a computationally effective procedure for finding out the interesting object $\Rightarrow$ do *not* provide analytic tools to establish if the given geodesic is globally maximal.

This is consequence of the *nonlocal* nature of the globally maximal curve — it cannot be identified by a *local* tool such as a differential equation.

Accessible tools: high symmetry and use of the Gaussian normal geodesic coordinate frame.