R-separable diagonal metrics in dimension four

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Example

R-separation of Helmholtz equation $(\Delta + k^2)\psi = 0$ (R.Prus, A.Sym)

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Helmholtz equation $(\Delta + k^2)\psi = 0$ on $\mathbb{R}^3$ with coordinates (u, v, w) such that metric is given by $(a, b, c = const.)$

$$ds^2 = \frac{b^2(w - a\cosh v)^2}{(a\cosh v - c\cos u)^2}du^2 + \frac{b^2(w - c\cos u)^2}{(a\cosh v - c\cos u)^2}dv^2 + dw^2.$$

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If

$$R = (a\cosh v - w)^{-1/2}(w - c\cos u)^{-1/2}$$

and the following system

$$\varphi_1'' + \frac{1}{4}\varphi_1 = 0, \qquad \varphi_2'' - \frac{1}{4}\varphi_2 = 0, \qquad \varphi_3'' + k^2\varphi_3 = 0.$$

holds then function

$$\psi = R(u, v, w)\varphi_1(u)\varphi_2(v)\varphi_3(w)$$

satifies Helmholtz equation.

Orthogonal coordinates $u = (u_1, u_2, \ldots, u_n)$ in n-dim. space $\mathcal{M}^n$

Assumption

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- $n = 2$: all metrics are conformally flat,
$$\mathbf{g} = \mathbf{g}_{ij} dx^i dx^j = H_1^2 du_1^2 + H_2^2 du_2^2 = \Omega(dx^2 + dy^2)$$

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- $n = 3$: there always exist orthogonal coordinates (G. Darboux, É. Cartan, D.M. Deturc & D. Yang):
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- $n \geq 4$: existence of orthogonal coordinates is an additional assumption

R-separability of Schrödinger equation in n-dim. space

Definition

The Schrödinger equation

$$\Delta\psi + (k^2 - V)\psi = 0 \qquad (1)$$

where

$$\Delta = h^{-1}\sum_{i=1}^{n}\partial_i \frac{h}{\epsilon_i H_i^2}\partial_i, \qquad h = H_1 H_2 \dots H_n, \qquad \epsilon_i = \pm 1$$

is R-separable (or metric $\mathbf{g} = \sum_{i=1}^{n}\epsilon_i H_i^2 du_i^2$ is R-separable in the Schrödinger eq.) if there exist $2n+1$ functions $R(u)$ and $p_i(u_i)$, $q_i(u_i)$ such that the following implication holds

$$\varphi_i'' + p_i\varphi_i' + q_i\varphi_i = 0 \quad (i = 1, 2, \dots, n)$$
$$\Downarrow$$
$$\psi(u) = R(u)\varphi_1(u_1)\varphi_2(u_2)\dots\varphi_n(u_n) \text{ solves } (1).$$

R-separability of Schrödinger equation in $\mathcal{R}^n$

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$$\Delta R - \left(k^2 - V - \sum_{i=1}^{n} \frac{1}{\epsilon_i H_i^2} q_i\right) R = 0.$$

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Binary metrics

The following class of metrics satisfies the first condition of R-separability:

$$\mathbf{g} = \frac{1}{R^2}\left[\frac{(u_1-u_2)^\gamma(u_1-u_3)^\gamma(u_1-u_4)^\gamma}{F_1(u_1)}du_1^2 + \frac{(u_1-u_2)^\gamma(u_2-u_3)^\gamma(u_2-u_4)^\gamma}{F_2(u_2)}du_2^2 \right.$$
$$\left. + \frac{(u_1-u_3)^\gamma(u_2-u_3)^\gamma(u_3-u_4)^\gamma}{F_3(u_3)}du_3^2 + \frac{(u_1-u_4)^\gamma(u_2-u_4)^\gamma(u_3-u_4)^\gamma}{F_4(u_4)}du_4^2\right]$$

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The Weyl tensor of the metric is given by

$$W_{ijkl} = 0, \quad W^k{}_{ikj} = 0, \quad W^{ij}{}_{ij} \neq 0.$$

The necessary condition for binary metrics to be conformally flat is given by

$$\gamma(\gamma-1)\sum_{i=1}^{4}\Big(\prod_{\substack{k<l\\k,l\neq i}}(u_k-u_l)^{\gamma+2}\Big)F_i=0.$$

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From the above equation we can prove the following

Fact. *If the binary metric* $\mathbf{g}$ *is conformally flat then*

$$\gamma\in\{-2,-1,0,1\}\,.$$

Conformally flat binary metrics:

- $\gamma = -2$

$$\mathbf{g} = \sum_{i=1}^{4} \frac{du_i^2}{F_i \prod_{j\neq i}(u_i - u_j)^2}, \qquad \sum_{i=1}^{4} F_i = 0,$$

- $\gamma = -1$

$$\mathbf{g} = \sum_{i=1}^{4} \frac{du_i^2}{\prod_{j\neq i}(u_i - u_j) \sum_{k=0}^{2} a_k u_i^k},$$

- $\gamma = 1$

$$\mathbf{g} = \sum_{i=1}^{4} \frac{\prod_{j\neq i}(u_i - u_j)}{\sum_{k=0}^{6} a_k u_i^k} du_i^2.$$

Binary metrics and Einstein equations

The following class of metrics satisfies the first condition of R-separability:

$$\mathbf{g} = \frac{(u_1 - u_2)^\gamma (u_1 - u_3)^\gamma (u_1 - u_4)^\gamma}{F_1(u_1)} du_1^2 + \frac{(u_1 - u_2)^\gamma (u_2 - u_3)^\gamma (u_2 - u_4)^\gamma}{F_2(u_2)} du_2^2 + \frac{(u_1 - u_3)^\gamma (u_2 - u_3)^\gamma (u_3 - u_4)^\gamma}{F_3(u_3)} du_3^2 + \frac{(u_1 - u_4)^\gamma (u_2 - u_4)^\gamma (u_3 - u_4)^\gamma}{F_4(u_4)} du_4^2 \; ,$$

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where $\gamma = const.$

Are there any Einstein metrics conformal to the binary metrics?

Is there a function Ω such that metric $\tilde{\mathbf{g}} = e^{2\Omega}\mathbf{g}$ satisfies Einstein equations with cosmological constant?

Bach and Cotton tensor are given by

$$B_{ij} = \nabla^k \nabla^l W_{iljk} + \frac{1}{2} R^{kl} W_{ikjl},$$

$$A_{ijk} = \frac{1}{n-3} \nabla_l W^l{}_{ijk}.$$

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Theorem (e.g. R. Gover & P. Nurowski)

If $\mathbf{g}$ is conformal to Einstein (i.e. metric $\tilde{\mathbf{g}} = e^{2\Omega}\mathbf{g}$ is a solution of Einstein equations with cosmological constant) then Cotton tensor C_{ijk} and Bach tensor B_{ij} satisfy the following conditions

$$A_{ijk} + \Omega^l W_{lijk} = 0,$$
$$B_{ij} + (n-4)\Omega^k \Omega^l W_{kijl} = 0$$

for some gradient $\Omega_i = \nabla_i \Omega$.

In dimension $n = 4$ the necessary condition for metric $\mathbf{g}$ to be conformal to Einstein is vanishing of Bach tensor,

$$B_{ij} = 0.$$

The following combination of off-diagonal components of the Bach tensor

$$c_{12}B_{12} + c_{13}B_{13} + c_{14}B_{14} + c_{23}B_{23} + c_{24}B_{24} + c_{34}B_{34} = 0,$$

where

$$c_{ij} = (-1)^{i+j}(u_i - u_j)^3(u_k - u_l), \qquad k < l,\ k, l \neq i, j$$

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$$\gamma(\gamma - 1)(2\gamma + 3)\sum_{i=1}^{4}\Big(\prod_{\substack{j<k \\ j,k\neq i}}(u_j - u_k)^{\gamma+2}\Big)F_i(u_i) = 0.$$

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Additional case

$$\gamma = -\frac{3}{2}\ .$$

Binary metrics and Einstein equations

The following metric

$$\mathbf{g} = \frac{(u_1-u_2)^\gamma(u_1-u_3)^\gamma(u_1-u_4)^\gamma}{a_1u_1+b_1}du_1^2 + \frac{(u_1-u_2)^\gamma(u_2-u_3)^\gamma(u_2-u_4)^\gamma}{a_2u_2+b_2}du_2^2 + \frac{(u_1-u_3)^\gamma(u_2-u_3)^\gamma(u_3-u_4)^\gamma}{a_3u_3+b_3}du_3^2 + \frac{(u_1-u_4)^\gamma(u_2-u_4)^\gamma(u_3-u_4)^\gamma}{a_4u_4+b_4}du_4^2 \,,$$

where $\gamma = -\frac{3}{2}$ and

$$\begin{aligned} a_1 &= B\alpha_1 - A\alpha_2, & b_1 &= B\beta_1 - A\beta_2 \\ a_2 &= -B\alpha_1 - A\alpha_2, & b_2 &= -B\beta_1 - A\beta_2 \\ a_3 &= A\alpha_1 + B\alpha_2, & b_3 &= A\beta_1 + A\beta_2 \\ a_4 &= A\alpha_1 - B\alpha_2, & b_4 &= A\beta_1 - A\beta_2, \end{aligned}$$

$(A, B, \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R})$ has vanishing Bach tensor $B_{ij} = 0$.

Conformal factor, $\tilde{\mathbf{g}} = e^{2\Omega}\mathbf{g}$

The conformal factor Ω satisfies the following system

$$\begin{aligned}
\partial_1\Omega &= -\partial_1(\log \mathbf{X}) + \frac{3}{4}\partial_1 \log\left[\mathrm{u}_{14}\mathrm{u}_{13}^2\right] + \frac{3}{4}\frac{\mathbf{Y}}{\mathbf{X}}\partial_1 \log\frac{\mathrm{u}_{13}}{\mathrm{u}_{14}}\\
\partial_2\Omega &= \dots,\\
\partial_3\Omega &= \dots,\\
\partial_4\Omega &= \dots.
\end{aligned}$$

where

$$\mathrm{u}_{ij} = u_i - u_j, \qquad \mathbf{X} = W^{12}{}_{12}, \qquad \mathbf{Y} = W^{13}{}_{13}.$$

$$
\begin{aligned}
\mathbf{X} = W^{12}{}_{12} = \frac{\gamma}{12}\Bigg[& (\mathrm{u}_{12}\mathrm{u}_{13}\mathrm{u}_{14})^{-\gamma-1}\left(\gamma(\mathrm{u}_{23}+\mathrm{u}_{24}) - \left(\frac{\mathrm{u}_{12}\mathrm{u}_{13}}{\mathrm{u}_{14}} + \frac{\mathrm{u}_{12}\mathrm{u}_{14}}{\mathrm{u}_{13}} - \frac{2\mathrm{u}_{13}\mathrm{u}_{14}}{\mathrm{u}_{12}}\right)\right) F_1 \\
& + (\mathrm{u}_{12}\mathrm{u}_{23}\mathrm{u}_{24})^{-\gamma-1}\left(-\gamma(\mathrm{u}_{13}+\mathrm{u}_{14}) - \left(\frac{\mathrm{u}_{12}\mathrm{u}_{23}}{\mathrm{u}_{24}} + \frac{\mathrm{u}_{12}\mathrm{u}_{24}}{\mathrm{u}_{23}} - \frac{2\mathrm{u}_{23}\mathrm{u}_{24}}{\mathrm{u}_{12}}\right)\right) F_2 \\
& + (\mathrm{u}_{13}\mathrm{u}_{23}\mathrm{u}_{34})^{-\gamma-1}\left(-\gamma(\mathrm{u}_{14}+\mathrm{u}_{24}) - \left(\frac{\mathrm{u}_{13}\mathrm{u}_{34}}{\mathrm{u}_{23}} \frac{\mathrm{u}_{23}\mathrm{u}_{34}}{\mathrm{u}_{13}} - \frac{2\mathrm{u}_{13}\mathrm{u}_{23}}{\mathrm{u}_{34}}\right)\right) F_3 + \\
& + (\mathrm{u}_{14}\mathrm{u}_{24}\mathrm{u}_{34})^{-\gamma-1}\left(\gamma(\mathrm{u}_{13}+\mathrm{u}_{23}) - \left(\frac{\mathrm{u}_{14}\mathrm{u}_{34}}{\mathrm{u}_{24}} + \frac{\mathrm{u}_{24}\mathrm{u}_{34}}{\mathrm{u}_{14}} - \frac{2\mathrm{u}_{14}\mathrm{u}_{24}}{\mathrm{u}_{34}}\right)\right) F_4 \\
& - \frac{1}{2}(\mathrm{u}_{13}\mathrm{u}_{24} + \mathrm{u}_{23}\mathrm{u}_{14})\Big((\mathrm{u}_{12}\mathrm{u}_{13}\mathrm{u}_{14})^{-\gamma-1}F_1' - (\mathrm{u}_{12}\mathrm{u}_{23}\mathrm{u}_{24})^{-\gamma-1}F_2' \\
& + (\mathrm{u}_{13}\mathrm{u}_{23}\mathrm{u}_{34})^{-\gamma-1}F_3' - (\mathrm{u}_{14}\mathrm{u}_{24}\mathrm{u}_{34})^{-\gamma-1}F_4'\Big)\Bigg],
\end{aligned}
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where

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\mathrm{u}_{ij} = u_i - u_j, \qquad \gamma = -\frac{3}{2}.
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$$
\begin{aligned}
\mathbf{Y} =& W^{13}{}_{13} = \frac{\gamma}{12}\Bigg[(\mathrm{u}_{12}\mathrm{u}_{13}\mathrm{u}_{14})^{-\gamma-1}\left(-\gamma(\mathrm{u}_{23}-\mathrm{u}_{34}) - \left(\frac{\mathrm{u}_{12}\mathrm{u}_{13}}{\mathrm{u}_{14}} - 2\frac{\mathrm{u}_{12}\mathrm{u}_{14}}{\mathrm{u}_{13}} + \frac{2\mathrm{u}_{13}\mathrm{u}_{14}}{\mathrm{u}_{12}}\right)\right)F_1 \\
&+ (\mathrm{u}_{12}\mathrm{u}_{23}\mathrm{u}_{24})^{-\gamma-1}\left(\gamma(\mathrm{u}_{14}+\mathrm{u}_{34}) - \left(-2\frac{\mathrm{u}_{12}\mathrm{u}_{23}}{\mathrm{u}_{24}} + \frac{\mathrm{u}_{12}\mathrm{u}_{24}}{\mathrm{u}_{23}} + \frac{\mathrm{u}_{23}\mathrm{u}_{24}}{\mathrm{u}_{12}}\right)\right)F_2 \\
&+ (\mathrm{u}_{13}\mathrm{u}_{23}\mathrm{u}_{34})^{-\gamma-1}\left(\gamma(\mathrm{u}_{12}+\mathrm{u}_{14}) - \left(\frac{\mathrm{u}_{13}\mathrm{u}_{34}}{\mathrm{u}_{23}} + \frac{\mathrm{u}_{13}\mathrm{u}_{23}}{\mathrm{u}_{34}} - \frac{2\mathrm{u}_{23}\mathrm{u}_{34}}{\mathrm{u}_{13}}\right)\right)F_3 + \\
&+ (\mathrm{u}_{14}\mathrm{u}_{24}\mathrm{u}_{34})^{-\gamma-1}\left(\gamma(\mathrm{u}_{12}-\mathrm{u}_{23}) - \left(\frac{\mathrm{u}_{14}\mathrm{u}_{14}}{\mathrm{u}_{34}} - \frac{\mathrm{u}_{14}\mathrm{u}_{34}}{\mathrm{u}_{24}} + \frac{\mathrm{u}_{24}\mathrm{u}_{34}}{\mathrm{u}_{14}}\right)\right)F_4 \\
&- \frac{1}{2}(\mathrm{u}_{12}\mathrm{u}_{34} - \mathrm{u}_{14}\mathrm{u}_{23})\Big((\mathrm{u}_{12}\mathrm{u}_{13}\mathrm{u}_{14})^{-\gamma-1}F_1' - (\mathrm{u}_{12}\mathrm{u}_{23}\mathrm{u}_{24})^{-\gamma-1}F_2' \\
&+ (\mathrm{u}_{13}\mathrm{u}_{23}\mathrm{u}_{34})^{-\gamma-1}F_3' - (\mathrm{u}_{14}\mathrm{u}_{24}\mathrm{u}_{34})^{-\gamma-1}F_4'\Big)\Bigg],
\end{aligned}
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where

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\mathrm{u}_{ij} = u_i - u_j, \qquad \gamma = -\frac{3}{2}.
$$

Conformal factor, $\tilde{\mathbf{g}} = e^{2\Omega}\mathbf{g}$

Using the fact that function $\frac{\mathbf{Y}}{\mathbf{X}}$ depends only on cross-ratio

$$s = \frac{(u_1 - u_4)(u_2 - u_3)}{(u_1 - u_3)(u_2 - u_4)},$$

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the system for Ω can be integrated. The solution is

$$e^{-\Omega} = (\mathbf{X} - \mathbf{Y})\, u_{13}^{-3/4}\, u_{14}^{-3/2}\, u_{23}^{-3/2}\, u_{24}^{\gamma/2}\, f(s)^{-1},$$

where

$$f(s) = A\left(2 - \frac{3}{\sqrt{s}}\right)\left(1 + \sqrt{s}\right)^{3/2} + B\left(2 + \frac{3}{\sqrt{s}}\right)\left(1 - \sqrt{s}\right)^{3/2}.$$

The end