Inhomogeneity effect in Wainwright-Marshman space-times

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Introduction

- *The 'fitting problem' in cosmology*, G. F. R. Ellis and W. Stoegert, CQG 1987
- *Inhomogeneity effects in Cosmology*, G. F. R. Ellis, 2011 and other contributions to the special issue of CQG
- *Gravitational Radiation in the Limit of High Frequency. II. Nonlinear Terms and the Effective Stress Tensor*, R. A. Isaacson, PR 1968
- *The high frequency limit in general relativity*, G. A. Burnett, JMP 1988
- *A new framework for analyzing the effects of small scale inhomogeneities in cosmology*, S. R. Green, R. M. Wald, PRD 2011
- *Inhomogeneity effect in Wainwright-Marshman space-times* , SJS, K. Głód, M. J. Wyrębowski, A. Konieczny, PRD 2014

Green–Wald framework

- $g_{ab}^{(0)}$ an averaged metric
- $h_{ab}(\lambda) = g_{ab}(\lambda) - g_{ab}^{(0)} << 1$
- $\lambda \to 0 \implies h_{ab} \to 0$
- assumptions
  1. for all $\lambda > 0$
     $$G_{ab}(g(\lambda)) + \Lambda g_{ab}(\lambda) = 8\pi T_{ab}(\lambda)\ ,$$
     where for all $\lambda > 0$ and all timelike vectors $t^a(\lambda)$ (with respect to $g_{ab}(\lambda)$)
     $$T_{ab}(\lambda) t^a(\lambda) t^b(\lambda) \geq 0$$
  2. $\exists$ a smooth positive function $C_1(x)$
     $$|h_{ab}(\lambda, x)| \leq \lambda C_1(x)$$
  3. $\exists$ a smooth positive function $C_2(x)$
     $$|\nabla_c h_{ab}(\lambda, x)| \leq C_2(x)$$
  4. $\exists$ a smooth tensor field μ_{abcdef} such that
     $$\underset{\lambda \to 0}{\text{w-lim}} \left(\nabla_a h_{cd}(\lambda) \nabla_b h_{ef}(\lambda) \right) = \mu_{abcdef}$$

Green–Wald framework

Let $A_{a_1 \dots a_n}(\lambda)$ be a one-parameter family of tensor fields defined for $\lambda > 0$. We say that $A_{a_1 \dots a_n}(\lambda)$ converges weakly to $B_{a_1 \dots a_n}$ as $\lambda \to 0$ if for all smooth $f^{a_1 \cdots a_n}$ of compact support, we have

$$\lim_{\lambda \to 0} \int f^{a_1 \dots a_n} A_{a_1 \dots a_n}(\lambda) = \int f^{a_1 \dots a_n} B_{a_1 \dots a_n}$$

The weak limit of the Einstein equations

$$G_{ab}(g^{(0)}) + \Lambda g^{(0)}_{ab} = 8\pi T^{(0)}_{ab} + 8\pi t^{(0)}_{ab}$$

$t^{(0)}_{ab}$ - contribution from inhomogeneities

Theorem (Green, Wald)

Given a one-parameter family $g_{ab}(\lambda)$ satisfying assumptions from the previous slide, the effective stress energy tensor $t^{(0)}_{ab}$ is traceless.

Green-Wald example (vacuum) — polarized Gowdy T^3 space-time

- *Examples of backreaction of small-scale inhomogeneities in cosmology*, S. Green, R. Wald, 2013
- the metric ($\alpha = \alpha(\tau, \theta)$, $P = P(\tau, \theta)$, $Q = Q(\tau, \theta)$)

$$
\begin{aligned}
g &= e^{(\tau-\alpha)}/2(-e^{-2\tau}d\tau^2 + d\theta^2) + e^{-\tau}[e^P d\sigma^2 + 2e^P Q d\sigma d\delta \\
&\quad +(e^P Q^2 + e^{-P})d\delta^2]
\end{aligned}
$$

- the example, J_k — Bessel functions
($Q = 0$, $N \to +\infty$ corresponds to $\lambda \to 0$, N - a discrete parameter)

$$
\begin{aligned}
P_N &= A/\sqrt{N} J_0(Ne^{-\tau})\sin(N\theta) \\
\alpha_N &= -\frac{A^2 e^{-\tau}}{2} J_1(Ne^{-\tau})J_0(Ne^{-\tau})\cos(2N\theta) - \frac{A^2 N e^{-2\tau}}{4} \\
&\quad \{[J_0(Ne^{-\tau})]^2 + 2[J_1(Ne^{-\tau})]^2 - J_0(Ne^{-\tau})J_2(Ne^{-\tau})\}
\end{aligned}
$$

our example — the Wainwright–Marshman solutions

- the metric (m - a constant, $n = n(u)$, $n = n(u)$, $u = t - z$),

$$g = t^{2m} e^{n}(-dt^2 + dz^2) + t^{1/2}\left(dx^2 + (t + w^2)dy^2 + 2w dx dy\right) \ ,$$

- the Einstein equations $n' = (w')^2$,
- the energy momentum tensor (the weak energy condition holds for $m \geq -3/16$)

$$\rho = p = \frac{1}{8\pi}(m + 3/16) t^{-2(m+1)} e^{-n}$$

- our example

$$w = \lambda \sin\frac{t - z}{\lambda} \ , \quad n = \frac{1}{2}\left(t - z + \frac{1}{2}\lambda \sin\frac{2(t - z)}{\lambda}\right) \ ,$$

An example — the Wainwright–Marshman solutions

- the background metric

$$g^{(0)} = t^{2m} e^{\frac{t-z}{2}} (-dt^2 + dz^2) + t^{1/2} \left(dx^2 + t dy^2 \right) \ ,$$

- the effective energy-momentum tensor

$$\begin{aligned} 8\pi t^{(0)}_{ab} &= \frac{1}{8} \left\{ -\mu^{c}{}_{c}{}^{de}{}_{de} - \mu^{c}{}_{c}{}^{d}{}_{d}{}^{e}{}_{e} + 2\mu^{cd}{}_{c}{}^{e}{}_{de} \right\} g^{(0)}_{ab} + \frac{1}{2} \mu^{cd}{}_{acbd} \\ &\quad - \frac{1}{2} \mu^{c}{}_{ca}{}^{d}{}_{bd} + \frac{1}{4} \mu_{ab}{}^{cd}{}_{cd} - \frac{1}{2} \mu^{c}{}_{(ab)c}{}^{d}{}_{d} \\ &\quad + \frac{3}{4} \mu^{c}{}_{cab}{}^{d}{}_{d} - \frac{1}{2} \mu^{cd}{}_{abcd} \, , \\ t^{(0)}_{tt} &= t^{(0)}_{zz} = -t^{(0)}_{tz} = -t^{(0)}_{zt} = \frac{1}{32\pi t} \end{aligned}$$

Summary

The problem of averaging is far from solved - but it is a problem that will not go away. Smaller scale inhomogeneities may possibly cause observable effects through dynamical backreaction, but this is a controversial suggestion. Ultimately, we probably need a general relativistic simulation of structure formation to resolve the issue of averaging.

G. F. R. Ellis