Dynamics of the Bianchi IX model near the cosmological singularity

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OUTLINE

Inspiration and challenge

Quantization of the Belinskii-Khalatnikov-Lifshitz scenario (1963-82).

- general and stable solution of GR that does not rely on any symmetry conditions; general corresponds to non-zero measure subset of all initial conditions; stable means stable against perturbation of initial conditions
- the best prototype for the BKL scenario is the non-diagonal Bianchi IX model
- obtaining quantum Bianchi IX model may enable quantization of the BKL theory

Metric of the Bianchi IX model

The general form of a line element of the non-diagonal Bianchi IX model, in the synchronous reference system, reads:

$$ds^2 = dt^2 - \gamma_{ab}(t) e^a_\alpha e^b_\beta dx^\alpha dx^\beta, \tag{1}$$

where $a, b, \ldots$ run from 1 to 3 and label frame vectors; $\alpha, \beta, \ldots$ take values $1, 2, 3$ and concern space coordinates, and where γ_{ab} is a spatial metric.

The homogeneity of the Bianchi IX model means that the three independent differential 1-forms $e^a_\alpha dx^\alpha$ are invariant under the transformations of the isometry group of the Bianchi IX model.

The cosmological time variable t is redefined as follows:

$$dt = \sqrt{\gamma}\, d\tau, \tag{2}$$

where γ denotes the determinant of γ_{ab}.

Equations of motion

Near the cosmological singularity:

1. the stress-energy tensor components can be ignored
2. the Ricci tensor components R_a^0 has negligible influence on the dynamics, which is well approximated by the R_0^0 and R_b^a components
3. one can use the Bianchi identities, freedom in the rotation of the metric γ_{ab} and frame vectors e_α^a, ignore rotation while keeping oscillation of the Kasner axes, and allow the anisotropy of space to grow without bound

Making use of these assumptions leads to specification of the dynamics[1].

[1] V. A. Belinskii, I. M. Khalatnikov and M. P. Ryan, "The oscillatory regime near the singularity in Bianchi-type IX universes", Preprint order **469** (1971), Landau Institute for Theoretical Physics, Moscow (unpublished); published as sections 1 and 2 in: M. P. Ryan, Ann. Phys. **70** (1971) 301.

Equations of motion (cont)

Finally, the asymptotic form (near the cosmological singularity) of the dynamical equations of the non-diagonal Bianchi IX model reads:

$$\frac{\partial^2 \ln a}{\partial \tau^2} = \frac{b}{a} - a^2, \quad \frac{\partial^2 \ln b}{\partial \tau^2} = a^2 - \frac{b}{a} + \frac{c}{b}, \quad \frac{\partial^2 \ln c}{\partial \tau^2} = a^2 - \frac{c}{b}. \tag{3}$$

The solutions to (3) must satisfy the condition:

$$\frac{\partial \ln a}{\partial \tau}\frac{\partial \ln b}{\partial \tau} + \frac{\partial \ln a}{\partial \tau}\frac{\partial \ln c}{\partial \tau} + \frac{\partial \ln b}{\partial \tau}\frac{\partial \ln c}{\partial \tau} = a^2 + \frac{b}{a} + \frac{c}{b}. \tag{4}$$

Eq (3) can be obtained from the Lagrangian equations of motion with L in the form:

$$L := \dot{x}_1\dot{x}_2 + \dot{x}_1\dot{x}_3 + \dot{x}_2\dot{x}_3 + \exp(2x_1) + \exp(x_2 - x_1) + \exp(x_3 - x_2). \tag{5}$$

Hamiltonian

The momenta, $p_I := \partial L / \partial \dot{x}_I$, are:

$$p_1 = \dot{x}_2 + \dot{x}_3, \quad p_2 = \dot{x}_1 + \dot{x}_3, \quad p_3 = \dot{x}_1 + \dot{x}_2. \tag{6}$$

The Hamiltonian of the system:

$$\begin{aligned} H \quad &:= p_I \dot{x}_I - L = \frac{1}{2}(p_1 p_2 + p_1 p_3 + p_2 p_3) \qquad (7) \\ &- \frac{1}{4}(p_1^2 + p_2^2 + p_3^2) - \exp(2x_1) - \exp(x_2 - x_1) - \exp(x_3 - x_2), \end{aligned}$$

which due to (6) and (4) leads to the dynamical constraint:

$$H = 0. \tag{8}$$

Hamilton's equations

The Hamilton equations have the following explicit form:

$$\dot{x}_1 = \frac{1}{2}(-p_1 + p_2 + p_3), \tag{9}$$

$$\dot{x}_2 = \frac{1}{2}(p_1 - p_2 + p_3), \tag{10}$$

$$\dot{x}_3 = \frac{1}{2}(p_1 + p_2 - p_3), \tag{11}$$

$$\dot{p}_1 = 2\exp(2x_1) - \exp(x_2 - x_1), \tag{12}$$

$$\dot{p}_2 = \exp(x_2 - x_1) - \exp(x_3 - x_2), \tag{13}$$

$$\dot{p}_3 = \exp(x_3 - x_2), \tag{14}$$

$$H = 0. \tag{15}$$

One may show that Lagrangian and Hamiltonian formulations are completely equivalent. Analytical solution to this 6-dimensional nonlinear coupled system of equations may not exist for any initial conditions (to be seen later).

Dynamical systems method

The local geometry of the phase space is characterized by the nature and position of its critical points. These points are locations where the derivatives of all the dynamical variables vanish. These are the points where phase trajectories may start, end, intersect, etc. The trajectories can also begin or end at infinities.

The set of finite and infinite critical points and their characteristic, given by the properties of the Jacobian matrix of the linearized equations at those points, may provide a qualitative description of a given dynamical system.

The above situation is specific to the case when a fixed point is of the hyperbolic type. In the case of the nonhyperbolic fixed point, linearized vector field at the fixed point cannot be used to specify completely local properties of the phase space.

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Dynamical systems analysis

Inserting $\dot{x}_1 = 0 = \dot{x}_2 = \dot{x}_3 = \dot{p}_1 = \dot{p}_2 = \dot{p}_3$ into l.h.s. of equations of motion and using the Hamiltonian constraint equation leads to:

$$p_1 = 0 = p_2 = p_3, \tag{16}$$

and

$$\begin{aligned} 0 &= \exp(2x_1), & (17)\\ 0 &= \exp(x_2 - x_1), & (18)\\ 0 &= \exp(x_3 - x_2). & (19) \end{aligned}$$

These conditions (17)-(19) are fulfilled for:

$$x_1 \to -\infty \quad x_2 \to -\infty,\ x_2 < x_1 < 0, \tag{20}$$

$$x_3 \to -\infty, \quad x_3 < x_2 < 0. \tag{21}$$

Dynamical systems analysis (cont)

Thus, the set of critical points fulfills the following conditions:

$$p_1 = 0 = p_2 = p_3, \tag{22}$$

$$x_1 \to -\infty, \; x_2 \to -\infty, \; x_3 \to -\infty, \tag{23}$$

$$x_3 < x_2 < x_1 < 0. \tag{24}$$

One may easily verify that this set satisfies the Hamiltonian constraint. Thus the set of critical points S_B is given by

$$\begin{aligned} S_B : \; = \; & \{(x_1, x_2, x_3, p_1, p_2, p_3) \in \bar{\mathbb{R}}^6 \mid (x_1 \to -\infty, \; x_2 \to -\infty, \; x_3 \to -\infty) \\ & \wedge (x_3 < x_2 < x_1 < 0); \;\; p_1 = 0 = p_2 = p_3\}, \end{aligned} \tag{25}$$

where $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.

Dynamical systems method (cont)

Stability properties are determined by the eigenvalues of the Jacobian of the system (9)-(14). More precisely, one has to linearize equations (9)-(14) at each point. Inserting $\vec{x} = \vec{x}_0 + \delta\vec{x}$, where $\vec{x} = (x_1, x_2, x_3, p_1, p_2, p_3)$, and keeping terms up to 1st order in $\delta\vec{x}$ leads to an evolution equation of the form $\delta\dot{\vec{x}} = J\delta\vec{x}$. Eigenvalues of J describe stability properties at the given point.

Dynamical systems analysis (cont)

The Jacobian J of the system reads:

$$J = \begin{pmatrix} 0 & 0 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial associated with Jacobian J reads: $P(\lambda) = \lambda^6$, so the eigenvalues are the following: $(0, 0, 0, 0, 0, 0)$. Since the real parts of all eigenvalues of the Jacobian are equal to zero, the set (25) consists of nonhyperbolic fixed points.

Summary

1. We are dealing with the nonhyperbolic type of critical points. Thus, getting insight into the structure of the space of orbits near such points requires an examination of the exact form of the vector field defining the phase space of our dynamical system. The information obtained from linearization is inconclusive.
2. The phase space is higher dimensional.
3. The set of critical points S_B is not a set of isolated points, but a 3-dimensional continuous subspace of $\bar{\mathbb{R}}^6$.
4. The critical subspace S_B is situated in an asymptotic region of phase space with infinite values of its variables.

Symplectic structure

In what follows we propose a new theoretical framework. We turn our system with Hamiltonian being a dynamical constraint into a new Hamiltonian system in which the Hamiltonian is no longer a constraint, but a generator of an evolution of the system. We call it the true Hamiltonian.

Let us rewrite the classical dynamics in terms of the true Hamiltonian. We consider the following factorization defined in the kinematical phase space:

$$\omega := \sum_{k=1}^{3} \left(dx_k \wedge dp_k \right) = \sum_{\alpha=1}^{2} \left(d\tilde{q}_\alpha \wedge d\tilde{\pi}_\alpha \right) + dt \wedge dH, \tag{26}$$

where $H \neq 0$ is an extension of H to the neighborhood of the constraint surface $H = 0$.

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Symplectic structure (cont)

We wish to map the symplectic structure (26) of the kinematical level into the physical phase space:

$$\Omega := \omega_{|_{H=0}} = \sum_{k=1}^{3} \Big(dx_k \wedge dp_k \Big)_{|_{H=0}}. \tag{27}$$

Suppose (27) can be rearranged to the following expression:

$$\Omega = \sum_{\alpha=1}^{2} \Big(dq_\alpha \wedge d\pi_\alpha \Big) + dT \wedge dH_T, \tag{28}$$

where T is dedicated to have the meaning of time, and where q_α, π_α and H_T are new canonical variables.
The existence of the expression (28) cannot be guaranteed in advance. However, let us assume that (28) can be constructed.

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Hamiltonian structure

Suppose that q_α and π_α are constants of motion (partial Dirac observables). Due to the factorization (28) we have:

$$\frac{d}{dT} q_\alpha := \{q_\alpha, H_T\}_{q,\pi} = \frac{\partial H_T}{\partial \pi_\alpha} \tag{29}$$

and

$$\frac{d}{dT} \pi_\alpha := \{\pi_\alpha, H_T\}_{q,\pi} = -\frac{\partial H_T}{\partial q_\alpha}, \tag{30}$$

where

$$\{\cdot, \cdot\}_{q,\pi} := \sum_{\alpha=1}^{2} \Big(\frac{\partial \cdot}{\partial q_\alpha} \frac{\partial \cdot}{\partial \pi_\alpha} - \frac{\partial \cdot}{\partial \pi_\alpha} \frac{\partial \cdot}{\partial q_\alpha} \Big). \tag{31}$$

Hamiltonian structure (cont)

Therefore, the existence of (28) implies the existence of the Hamiltonian structure of the reduced system in terms of Dirac's observables.

The dynamics is generated by the true Hamiltonian H_T and is parameterized by an evolution parameter T.

In this new setting the system has no dynamical constraints. One may examine it using the dynamical systems methods. The phase space is now only four dimensional, which simplifies analysis of the original dynamics defined in six dimensional phase space with the dynamical constraint.

The factorization procedure described above make sense outside the subspace of the critical points S_B defined by (25).

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Conclusions

- The nonhyperbolicity of critical points is generic for our system. It may mean bifurcation or chaotic behavior. But chaos underlies the BKL dynamics. We would probably worry if the results would be different.
- Parameterizing the phase space by the Dirac observables is highly promising. The physical phase space is only four dimensional. We expect that the subspace of critical points is lower dimensional so it can be examined by standard methods. Work is in progress!
- Independently we analyze in the same manner the diagonal Bianchi IX model to have framework for a more sophisticated non-diagonal model.
- After understanding the mathematical structure of the physical phase space one can begin preparation to quantization of the classical dynamics, first the diagonal model then hopefully non-diagonal Bianchi IX.