Cosmology of a diffusing fluid

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Classical equations

The Einstein equations ($g^{\mu\nu}$ is the metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$)

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G T^{\mu\nu}, \tag{1}$$

with

$$(T^{\mu\nu})_{;\mu} = 0. \tag{2}$$

RHS of Einstein equations

- phase space distribution of particles
- fields
- fluids

Phase space distribution

If particle's dynamics is determined by classical evolution equations, then the conservation law for the energy momentum is a consequence of the Liouville equation (where $\Gamma^{\mu}_{\nu\rho}$ are Christoffel symbols)

$$(p^{\mu}\partial^{x}_{\mu} - \Gamma^{k}_{\mu\nu}p^{\mu}p^{\nu}\partial_{k})\Phi(x,p) = 0. \tag{3}$$

The formula for the energy-momentum tensor

$$T^{\mu\nu} = \sqrt{g}\int \frac{d\mathbf{p}}{(2\pi)^{3}}\frac{1}{p^{0}}p^{\mu}p^{\nu}\Phi, \tag{4}$$

define the Einstein-Liouville-Vlasov system
g is the determinant of the metric and p^0 is determined from the mass-shell condition $p_{\mu}p^{\mu} = m^2$ (m is the particle's mass, we set $c = 1$). Greek indices run from 0 to 3, Latin indices denoting spatial components have the range from 1 to 3, the covariant derivative is over the space-time, derivatives over the momenta $\frac{\partial}{\partial p^k}$ are denoted ∂_k and ∂^x denotes a derivative over a space-time coordinate x.

Fluids

Assuming we have a phase space distribution we can define

$$v^{\mu} = \langle p^{\mu} \rangle \tag{5}$$

Then,

$$\langle p^{\mu} p^{\nu} \rangle = \langle 1 \rangle u^{\mu} u^{\nu} + \langle (p^{\mu} - u^{\mu})(p^{\nu} - u^{\nu}) \rangle \tag{6}$$

This can be expressed as

$$T^{\mu\nu} = E u^{\mu} u^{\nu} - \pi (g^{\mu\nu} - u^{\mu} u^{\nu}) + \pi^{\mu\nu} \tag{7}$$

where

$$g_{\mu\nu} u^{\mu} u^{\nu} = 1 \tag{8}$$

Fluids are applied to describe structure formation (we need matter)

Fields

If we have the action W then

$$T_{\mu\nu} = \frac{\delta W}{\delta g^{\mu\nu}(x)} \tag{9}$$

For the scalar field

$$W = \int dx \sqrt{g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)) \tag{10}$$

Classical scalar fields are applied to generate inflation

What if the energy-momentum is not conserved?

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = T^{\mu\nu} = T_D^{\mu\nu} + \tilde{T}^{\mu\nu}, \tag{11}$$

where T_D is the energy-momentum of a certain (dark) matter and $\tilde{T}$ is the energy-momentum of the system of diffusing particles. From the lhs it follows that

$$(T_D^{\mu\nu})_{;\mu} = -(\tilde{T}^{\mu\nu})_{;\mu}. \tag{12}$$

Knowing the rhs of we can determine the lhs up to a constant. We represent T_D by a time-dependent cosmological term Λ. A dynamical relation of the cosmological term to the matter density seems to be unavoidable for an explanation of the coincidence problem.

Why diffusion?

- **Diffusion equilibrates to a temperature (equilibrium) state washing out initial conditions**

The diffusion on the mass-shell

$$g_{\mu\nu} p^{\mu} p^{\nu} = m^2 \tag{13}$$

The diffusion is generated by the Laplace-Beltrami operator on $\mathcal{H}_+$

$$\triangle_H = \frac{1}{\sqrt{G}} \partial_j G^{jk} \sqrt{G} \partial_k \tag{14}$$

where

$$G^{jk} = m^2 g^{jk} + p^j p^k \tag{15}$$

$\partial_j = \frac{\partial}{\partial p^j}$ and $G = \det(G_{jk})$ is the determinant of G_{jk}.

The transport equation for the linear diffusion generated by $\triangle_H$ reads

$$(p^\mu \partial^x_\mu - \Gamma^k_{\mu\nu} p^\mu p^\nu \partial_k)\Omega = \kappa^2 \triangle_H \Omega, \tag{16}$$

where κ^2 is the diffusion constant, $\partial^x_\mu = \frac{\partial}{\partial x^\mu}$ and $x = (t, \mathbf{x})$

Quantum phase space distributions

If the phase space distribution has the Bose-Einstein or Fermi-Dirac equilibrium limit which is a minimum of the relative entropy (related to the free energy) then the diffusion equation must be non-linear. The proper generalization reads

$$(p^{\mu}\partial_{\mu}^{x}-\Gamma_{\mu\nu}^{k}p^{\mu}p^{\nu}\partial_{k})\Omega=\kappa^{2}p_{0}\partial_{j}\Big(G^{jk}p_{0}^{-1}\partial_{k}\Omega+\beta p^{j}\Omega(1+\nu\Omega)\Big), \tag{17}$$

where $\nu=1$ for bosons and $\nu=-1$ for fermions. The classical (Boltzmann) statistics can be described by $\nu=0$.

Solutions of linear and non-linear diffusion equations at finite temperature

We have the time-dependent equilibrium

$$\Omega_E^{PL} = \Big(\exp(\beta a^2(p+\mu)) - \nu\Big)^{-1} \tag{18}$$

where μ is an arbitrary constant (the chemical potential). In the ultrarelativistic limit (a large p) the Planck distribution is the same as the Jüttner distribution.
For the equilibrium solution we obtain standard Friedmann cosmology.
There are other solutions of the diffusion equation whose energy momentum tensor gives different Friedmann equation for the scale factor a

We solve the conservation equation for Λ then with $H = a^{-1}\frac{da}{d\tau}$ the Friedmann equation reads

$$\frac{3}{8\pi G}H^2 \equiv \frac{3}{8\pi G}(a^{-1}\frac{da}{d\tau})^2 = \tilde{T}^{00}(\tau) - \int_{\tau_0}^{\tau} dr a^{-4}\partial_r(a^4\tilde{T}^{00}) + \frac{\Lambda}{8\pi G}(\tau_0)$$
$$= \tilde{T}^{00}(\tau_0) - 4\int_{\tau_0}^{\tau} drH(r)\tilde{T}^{00}(r) + \frac{\Lambda}{8\pi G}(\tau_0) \tag{19}$$

I assume: $\tilde{T}^{\mu\nu}$ energy-momentum of diffusing particles,
$T_D^{\mu\nu} = \Lambda g^{\mu\nu}$
and (τ is the cosmic time)
$ds^2 = d\tau^2 - a^2(\tau)d\mathbf{x}^2$

Explicit solution

We can find an explicit power-like solution of the integro-differential equation by a fine tuning of parameters showing that the exponential behaviour is not a necessity even if $\Lambda(\tau_0) > 0$. Let us assume

$$a(\tau) = \nu(\tau - q)^{\gamma} \tag{20}$$

with the initial condition $a(\tau_0) = \nu(\tau_0 - q)^{\gamma}$. Inserting we determine the parameters

$$\gamma = 1, \tag{21}$$

$$\nu = \sigma^{\frac{1}{3}}, \tag{22}$$

$$(\tau_0 - q)^2 = \frac{2\theta}{\nu} \tag{23}$$

Then

$$\Lambda(\tau_0) = \frac{3}{2}(\tau_0 - q)^{-2}. \tag{24}$$

We obtain

$$\Lambda = 8\pi G\tilde{E} = \frac{3}{2}(\tau - q)^{-2} \tag{25}$$

Non-homogeneous metric:Fluctuation spectrum

$$\langle \frac{\delta T}{T}(\mathbf{n}) \frac{\delta T}{T}(\mathbf{n}') \rangle = \sum_{l=0}^{\infty} (2l+1) C_l P_l(\mathbf{n}\mathbf{n}') \qquad (26)$$

n direction in the sky

Experimental result: COBE, WMAP

$$l(l+1)C_l \simeq const \qquad (27)$$

ordinary Sachs-Wolfe effect

Einstein-Liouville-Vlasov equations

We decompose

$$g_{\mu\nu} = \overline{h}_{\mu\nu} + h_{\mu\nu} \tag{28}$$

where $\overline{h}_{\mu\nu}$ describes homogenous metric in the conformal time

$$ds^2 = \overline{h}_{\mu\nu} dx^\mu dx^\nu = a^2(dt^2 - d\mathbf{x}^2) \tag{29}$$

and

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2\Big((1 + 2\phi)dt^2 - (1 + 2\psi)d\mathbf{x}^2 - \gamma_{ij} dx^i dx^j\Big) \tag{30}$$

We write the Liouville equation in the form

$$(p^{\mu}\partial_{\mu}^{x} - \overline{\Gamma}_{\mu\nu}^{k} p^{\mu} p^{\nu} \partial_k)\Omega = \delta\Gamma_{\mu\nu}^{k} p^{\mu} p^{\nu} \partial_k \Omega \qquad (31)$$

For massless particles ($m = 0$) and in the homogeneous metric $h_{\mu\nu} = 0$ the Jüttner distribution

$$\Omega_E = \exp(-a^2\beta|\mathbf{p}|) \tag{32}$$

with

$$\mathbf{p}^2 = \sum_j p^j p^j$$

is the solution of Liouville eq.

Non-homogeneous metric

The solution can be expressed as

$$\Omega = \Omega_E^g + \beta p_0 \Theta \Omega_E^g \tag{33}$$

where

$$\Omega_E^g = \exp(-\beta p_0) \tag{34}$$

and p_0 is determined from $g_{\mu\nu} p^\mu p^\nu = 0$

Θ is the solution of the equation

$$\partial_t \Theta + n^k \partial_k^x \Theta = -2\partial_t \psi - \frac{1}{2} n^j n^k \partial_t \gamma_{jk} \tag{35}$$

We have

$$\Omega = \exp(-\frac{p_0}{T+\delta T}) \tag{36}$$

where

$$\frac{\delta T}{T} = \Theta \tag{37}$$

where

$$\Theta(t,\mathbf{x}) = \Theta_0(\mathbf{x}-\mathbf{n}t) - \int_0^t \Big(2\partial_s\psi(s,\mathbf{x}-(t-s)\mathbf{n}) + \tfrac{1}{2}\partial_s\gamma_{jk}(s,\mathbf{x}-(t-s)\mathbf{n})n^j n^k\Big) ds \tag{38}$$

with the initial condition $\Theta_0(\mathbf{x})$.

Diffusive temperature fluctuations

We write

$$\Omega = \Omega_E(1 + \chi) \tag{39}$$

We consider the massless (ultrarelativistic) limit $m = 0$.

$$\begin{aligned}
&\partial_t\chi + n^k\partial_k^x\chi - 2\mathcal{H}p^k\partial_k\chi - \kappa^2 p^{-1}\triangle_H^0\chi + \kappa^2\beta a^2 p^k\partial_k\chi \\
&= -a^2\beta|\mathbf{p}|(n^k\partial_k^x(\phi+\psi) + 2\partial_t\psi \\
&+\tfrac{1}{2}n^l\partial_l\gamma_{jk}n^j n^k + \partial_t\gamma_{jk}n^j n^k)
\end{aligned} \tag{40}$$

where in the massless case the operator $\triangle_H$ can be expressed in the form

$$\triangle_H^0 = p^j p^k\partial_j\partial_k + 3p^k\partial_k, \tag{41}$$

We look for solutions in the form

$$\Omega = \Omega_E^g + \beta p_0 \Theta \Omega_E^g + ra^2 \Omega_E^g = (1 + ra^2) \exp(-\frac{p_0}{T + \delta T}) \quad (42)$$

Inserting this formula in the diffusion equation we obtain equations for the temperature fluctuation Θ and r

$$\partial_t \Theta + n^k \partial_k^x \Theta + \kappa^2 \beta a^2 \Theta = -2\partial_t \psi - \frac{1}{2} n^j n^k \partial_t \gamma_{jk} \quad (43)$$

$$\partial_t r + n^k \partial_k^x r = 3\kappa^2 \Theta \quad (44)$$

where

$$n^k = p^k |\mathbf{p}|^{-1} \quad (45)$$

The solution reads

$$\Theta_t(\mathbf{x}) = \exp(-\beta\kappa^2 \int_0^t a^2(s)ds)\Theta_0(\mathbf{x} - t\mathbf{n}) \\ - \int_0^t ds \exp(-\beta\kappa^2 \int_s^t a^2(r)dr)(2\partial_s\psi(s, \mathbf{x} - (t-s)\mathbf{n}) \\ + \frac{1}{2}\partial_s\gamma_{jk}n^j n^k(s, \mathbf{x} - (t-s)\mathbf{n}) \tag{46}$$

Temperature fluctuations

We restrict ourselves to tensor perturbations

$$\langle\Theta(t,\mathbf{n})\Theta(t,\mathbf{n}')\rangle =$$
$$= \frac{1}{4}(2\pi)^{-3}\int_0^t ds\int_0^t ds'\int d\mathbf{q}F(s,s',q)\exp(-\beta\kappa^2(\int_s^t+\int_{s'}^t)dra^2(r))$$
$$(2(\mathbf{n}\Delta(\mathbf{q})\mathbf{n}')^2-(\mathbf{n}\Delta(\mathbf{q})\mathbf{n})(\mathbf{n}'\Delta(\mathbf{q})\mathbf{n}'))\exp(-i(t-s)\mathbf{nq}+i(t-s')\mathbf{n}'\mathbf{q}) \tag{47}$$

where

$$\mathbf{n}\Delta(\mathbf{q})\mathbf{n}' = \mathbf{n}\mathbf{n}' - \mathbf{q}^{-2}(\mathbf{q}\mathbf{n})(\mathbf{q}\mathbf{n}') \equiv \Delta(\mathbf{n}\mathbf{n}', \mathbf{e}\mathbf{n}, \mathbf{e}\mathbf{n}') \tag{48}$$

$$\mathbf{n}\Delta(\mathbf{q})\mathbf{n} = 1 - \mathbf{q}^{-2}(\mathbf{q}\mathbf{n})^2 \equiv \Delta(\mathbf{e}\mathbf{n}) \tag{49}$$

where we write $\mathbf{q} = q\mathbf{e}$ and

$$F(s, s', q) = \partial_s \partial_{s'} P(s, s', \mathbf{q}) \tag{50}$$

P is the expectation value of tensor perturbations.

If the power spectrum F is known then there remains to perform the integrals over s and q in order to obtain

$$\begin{aligned}&\langle\Theta(t,\mathbf{n})\Theta(t,\mathbf{n}')\rangle == \sum_{l=0}^{\infty}(2l+1)\tilde{D}_l(t,\mathbf{nn}')P_l(\mathbf{nn}')\\ &= \sum_{l=0}^{\infty}(2l+1)C_l(t)P_l(\mathbf{nn}')\end{aligned} \tag{51}$$

where P_l are the Legendre polynomials and $\tilde{D}_l P_l$ still must be expanded in Legendre polynomials if the coefficients C_l are to be independent of the angles. We have

$$\begin{aligned}&\tilde{D}_l == \frac{1}{16\pi^2}\int_0^t ds\int_0^t ds'\int d\mathbf{q}F(s,s',q)\exp(-\beta\kappa^2(\int_s^t+\int_{s'}^t)dra^2(r))\\ &\Big(2\Delta(\mathbf{nn}',-i\partial_s,i\partial_{s'})^2-\Delta(-i\partial_s)\Delta(i\partial_{s'})\Big)j_l(q(t-s))j_l(q(t-s'))\end{aligned} \tag{52}$$

Let us calculate only the first term (denoted D_l)

$$D_l = \frac{1}{8\pi^2}(\mathbf{nn}')^2 \int_0^t ds' \int_0^t ds \exp(-\beta\kappa^2(\int_s^t + \int_{s'}^t) dr a^2(r)) \int_0^\infty dq q^2 F(s,s',q) j_l(q(t-s)) j_l(q(t-s')) \quad (53)$$

It gives the same behaviour at large l as C_l. We make a simplifying assumption

$$F(s,s',q) = f(s,s')\sigma(q) \quad (54)$$

We shall first estimate D_l for a large l. According to Limber asymptotic formula for large l

$$D_l = \frac{1}{16\pi}(\mathbf{nn}')^2 \int_0^t ds (t-s)^{-2} f(s,s) \exp(-2\beta\kappa^2 \int_s^t dr a^2(r)) \sigma(\frac{l+\frac{1}{2}}{t-s})(1+O(l^{-2}))) \quad (55)$$

Then, assuming

$$\sigma(q) = Aq^{-3} \tag{56}$$

we estimate the asymptotic behaviour

$$D_l = A(l + \frac{1}{2})^{-3} \frac{1}{16\pi} (\mathbf{nn}')^2 \int_0^t ds (t-s) f(s,s) \exp(-2\beta\kappa^2 \int_s^t dr a^2(r)) \tag{57}$$

We can obtain an exact result

$$D_l(t) = \frac{1}{8\pi} A(\mathbf{nn}')^2 \frac{1}{2\sqrt{\pi}} \frac{(l-1)!}{\Gamma(l+\frac{3}{2})} \int_0^t ds \int_0^s ds' (\frac{t-s'}{t-s})^l$$
$$f(s,s') \exp\Big(-\beta\kappa^2 (\int_s^t + \int_{s'}^t) dr a^2(r) \Big) F(l, -\frac{1}{2}, l + \frac{3}{2}, \frac{(t-s')^2}{(t-s)^2}) \tag{58}$$

If the integral over s and s' is concentrated at $s = s'$,e.g.,

$$f(s,s') = f_d \delta(s - s_d) \delta(s' - s_d)$$

then we can set $s = s'$ and using

$$\int_0^\infty dq q^{-2} J_{l+\frac{1}{2}}(\gamma q) J_{l+\frac{1}{2}}(\gamma q) = \frac{\gamma}{\pi} \frac{(l-1)!}{(l+1)!} \tag{59}$$

we obtain

$$D_l(t) = A f_d (\mathbf{n}\mathbf{n}')^2 \frac{1}{8\pi} (l(l+1))^{-1} \exp(-2\beta\kappa^2 (\int_{s_d}^t dr a^2(r)) \tag{60}$$

For a finite temperature the metric fluctuations have the form

$$F(s,s',q) = q(\exp(\beta q) - 1)^{-1} \exp(-i(s-s')q) \qquad (61)$$

Its low and high energy behaviour can be described by the power spectrum

$$F(s,s',q) = Qq^{n-1} \exp(-\beta q - i(s-s')q) \qquad (62)$$

$n = 1$ describes the behaviour of thermal correlation functions of $\partial_t \gamma$ for a small q and $n = 2$ for a large q. A power spectrum with an arbitrary power n may result from a accelerated expansion starting from a thermal state. The Limber formula leads to the large l behaviour in the Planck case

$$D_l(t) = (l+\tfrac{1}{2})\tfrac{1}{16\pi}(\mathbf{nn}')^2 \int_0^t ds(t-s)^{-3}\Big(\exp(\tfrac{\beta(l+\frac{1}{2})}{t-s})-1\Big)^{-1}$$
$$\exp(-\beta\kappa^2 \int_s^t dra^2(r)) \qquad (63)$$

For the power spectrum we have

$$D_l(t) = (l+\tfrac{1}{2})^{n-1}\tfrac{1}{16\pi}Q(\mathbf{nn}')^2 \int_0^t ds(t-s)^{-1-n}$$
$$\exp\Big(-\tfrac{\beta(l+\frac{1}{2})}{t-s}-\beta\kappa^2\int_s^t dra^2(r)\Big) \qquad (64)$$

The integral could be calculated by means of the saddle point method. The saddle point is determined from the equation

$$-\frac{1+n}{t-s_c}+\beta\frac{l+\frac{1}{2}}{(t-s_c)^2}=\beta\kappa^2 a^2(s_c) \qquad (65)$$

The dependence of D_l on l is a function of the expansion scale a and the dissipation rate κ^2.

Graviton correlation functions:power spectrum

We expand solution of Einstein equations in an external homogeneous metric in plane waves

$$\gamma_{jk}(t,\mathbf{x}) = (2\pi)^{-\frac{3}{2}} \int d\mathbf{q}|\mathbf{q}|^{-\frac{1}{2}}(a(\lambda,\mathbf{q})e_{jk}(\lambda,\mathbf{q})\gamma(t,\mathbf{q},\lambda)\exp(i\mathbf{q}\mathbf{x})+cc) \tag{66}$$

where cc denotes the complex conjugation of the preceding term). Quantizing the Fourier modes we obtain the vacuum correlation functions of gravitons in the transverse-traceless gauge. So, in the momentum space

$$\langle\gamma_{jn}(t,\mathbf{k})\gamma_{lr}(t',\mathbf{q})\rangle = \delta(\mathbf{k}+\mathbf{q})(2\pi)^{-3}P(t,t',\mathbf{q})(\Delta_{jl}\Delta_{rn}+\Delta_{jr}\Delta_{nl}-\Delta_{jn}\Delta_{lr}) \tag{67}$$

where

$$\Delta_{jl} = \delta_{jl} - q_j q_l \mathbf{q}^{-2} \tag{68}$$

and

$$P(t,t',q) = \sum_{\lambda,k,j} \overline{e_j^k(\lambda,\mathbf{q})\gamma(t,\mathbf{q},\lambda)} e_j^k(\lambda,\mathbf{q})\gamma(t',\mathbf{q},\lambda) \qquad (69)$$

P depends only on $q = |\mathbf{q}|$ because the modes $\gamma(t,q,\lambda)$ depend only on q. Taking the time derivatives of the metric

$$\langle \partial_t \gamma_{jn}(t,\mathbf{k}) \partial_{t'} \gamma_{lr}(t',\mathbf{q}') \rangle = \delta(\mathbf{k}+\mathbf{q})(2\pi)^{-3} F(t,t',\mathbf{q})(\Delta_{jl}\Delta_{rn} + \Delta_{jr}\Delta_{nl} - \Delta_{jn} \qquad (70)$$

where

$$F(t,t',\mathbf{q}) = \partial_t \partial_{t'} P(t,t',\mathbf{q}) \qquad (71)$$

The power spectrum is closely related to the energy ϵ_g of gravitational waves [**?**]

$$\epsilon_g = \frac{a^2}{32\pi G} \int d\mathbf{q} F(t,t,q) \qquad (72)$$

We can understand the power spectrum appearing in temperature fluctuations in a wider sense as the energy impulse coming from gravitational waves and leading to the temperature fluctuations. We could obtain such generalized F calculating the correlation functions of γ in more general states than just the vacuum states. In such a case the impulse described by F can be restricted in time.

We restrict ourselves to standard examples of known vacuum correlation functions. First, for the de Sitter space in the metric $ds^2 = t^{-2}(dt^2 - d\mathbf{x}^2))$

$$P_{dS}(t,t',\mathbf{q}) = a(t)a(t')\frac{1}{2|\mathbf{q}|}\Big(1+\frac{i(t-t')}{|\mathbf{q}|tt'}+(\mathbf{q}^2tt')^{-1}\Big)\exp(-i(t-t')|\mathbf{q}|) \tag{73}$$

We approximate for a small q

$$\partial_s\partial_{s'}P_{dS}(s,s',\mathbf{q}) \simeq \frac{1}{2}|\mathbf{q}|^{-3}(ss')^{-2}\exp(-i(s-s')|\mathbf{q}|) \tag{74}$$

Next, correlation functions in Minkowski space at temperature β^{-1} have the form

$$P_\beta(t,t',\mathbf{q}) = \frac{1}{2|\mathbf{q}|}(\exp(\beta|\mathbf{q}|)-1)^{-1}\exp(-i(t-t')|\mathbf{q}|) \tag{75}$$

We can make the approximation

$$\partial_s \partial_{s'} P_\beta(s, s', \mathbf{q}) = \frac{1}{2}|\mathbf{q}| \exp(-\beta|\mathbf{q}|) \exp(-i(s - s')|\mathbf{q}|) \qquad (76)$$