Invariant Solutions of the Wheeler-DeWitt equation in Hybrid Gravity

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Field equations

The action of the hybrid metric-Palatini gravity is

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^4x \sqrt{-g}[R + f(\mathcal{R})] + S_m, \tag{1}$$

where R is a metric Ricci curvature scalar and $f(\mathcal{R})$ is a function of the Palatini curvature scalar which is constructed by an independent torsionless connection $\hat{\Gamma}$. Here $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}(\hat{\Gamma})$.
The modified field equations are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + f'(\mathcal{R})\mathcal{R}_{\mu\nu} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{2}$$

The trace of (2) is called the hybrid structural equation. The Palatini curvature $\mathcal{R}$ can be expressed algebraically in terms of X, assuming that $f(\mathcal{R})$ has analytic solutions:

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa^2 T + R \equiv X. \tag{3}$$

Field equations

The action (1) is equivalent to the following one with the scalar field

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}[R+\phi\mathcal{R}-V(\phi)]. \quad (4)$$

where $\phi \equiv f'(\mathcal{R})$ and $V(\phi) = \mathcal{R}f'(\mathcal{R}) - f(\mathcal{R})$. Furthermore, for the two tensors $R_{\mu\nu}$ and $\mathcal{R}_{\mu\nu}$ it holds that[1]

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + \frac{3}{2}\frac{f(\mathcal{R})_{,\mu}f(\mathcal{R})_{,\nu}}{f^2(\mathcal{R})} - \frac{f(\mathcal{R})_{;\mu\nu}}{f(\mathcal{R})} - \frac{1}{2}\frac{\Box f(\mathcal{R})}{f(\mathcal{R})}g_{\mu\nu}$$

[1] S. Capozziello, T. Harko, T.S. Koivisto, F.S.N. Lobo, G.J Olmo, JCAP 04 (2013) 011 (arXiv:1209.2895)

Field equations

By using the last relation the action (4) becomes

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^4x \sqrt{-g} [(1+\phi)R + \frac{3}{2\phi} \partial^\mu \phi \partial_\mu \phi - V(\phi)] \tag{5}$$

which is the action of a non minimally coupled scalar field.
For the FRW spatially flat spacetime and for empty space ($T_{\mu\nu} = 0$) the Lagrangian of the field equations is

$$\mathcal{L} = 6a\dot{a}^2(1+\phi) + 6a^2\dot{a}\dot{\phi} + \frac{3}{2\phi} a^3 \dot{\phi}^2 + a^3 V(\phi). \tag{6}$$

where the field equations are the Hamiltonian of (6) and the Euler-Lagrange equations with respect to the variables $x^i = (a, \phi)$.
The WDW equation which is a quantization of the Hamiltonian has a form $\Delta \Psi - a^3 V(\phi) \Psi = 0$, where Δ is the Laplace operator.

Point Symmetries

Let $H\left(x^i, u, u_{,i}, u_{,ij}\right) = 0$ be a PDE while $X = \xi^i\left(x^j, u\right)\partial_i + \eta\left(x^j, u\right)\partial_u$ is the generator of an infinitesimal transformation in the space $\left\{x^i, u\right\}$. We shall say that X is a Lie point symmetry of H if there exists a function λ, such that $X^{[2]}H = \lambda H\ ,\ \ mod H = 0$ where $X^{[2]}$ is the second prologation of X.

In order to select potentials $V\left(\phi\right)$ where the WDW eq. admits Lie point symmetries we will follow the geometric approach of A. Paliathanasis, M. Tsamparlis, IJGMMP (2014) 14500376, (arXiv:1312.3942) where Lie point symmetries are related to the conformal algebra of a minisuperspace.

Results

- If $V(\phi) = V_0\left(\sqrt{\phi} + V_1\right)^4$, the generic symmetry vector is

$$X_\Psi = -\frac{1}{2}\partial_a + \frac{\phi + V_1\sqrt{\phi}}{a}\partial_\phi + c_2\Psi\partial_\Psi$$

$$f(\mathcal{R}) = \frac{\mathcal{R}^2}{4V_0} \quad \text{for} \quad V_1 = 0.$$

- If $V(\phi) = V_0(1+\phi)^2 \exp\left(\frac{6}{V_1}\arctan\sqrt{\phi}\right)$, the generic symmetric vectors are $X_1 = \partial_u$, $X_\Psi = \Psi\partial_\Psi$
$X_2 = e^{-\frac{3v}{V_1}}\left[\cos(V_C u)\cos(3v) + \sin(V_C u)\sin(3v)\right]\partial_u +$
$e^{-\frac{3v}{V_1}}\left[\begin{array}{c} (V_1\cos(3v) - \sin(3v))\cos(V_C u) + \\ + (\cos(3v) + V_1\sin(3v))\sin(V_C u) \end{array}\right]\partial_v$
$X_3 = e^{-\frac{3v}{V_1}}\left[\cos(V_C u)\sin(3v) + \sin(V_C u)\cos(3v)\right]\partial_u$
$+e^{-\frac{3v}{V_1}}\left[\begin{array}{c} (\cos(3v) + V_1\sin(3v))\cos(V_C u) + \\ + (\sin(3v) - V_1\cos(3v))\sin(V_C) \end{array}\right]\partial_v$

For the power law potential $V_0\left(\sqrt{\phi}+V_1\right)^4$ there exist a coordinate system $(a,\phi)\rightarrow(x,y)$ where the WDW becomes

$$\Psi_{,xx}+\Psi_{,yy}-2V_0y^4\Psi=0.$$

Thus by applying the zero order invariants of X_Ψ, $\left\{y, Ye^{\mu x}\right\}$ we can find the invariant solution

$$\Psi\left(x,y\right)=\sum_{\mu}\left[y_1e^{\mu x+w(y)}+y_2e^{\mu x-w(y)}\right] \tag{7}$$

where $w\left(y\right)=\frac{\sqrt{2}}{2}\int\sqrt{\left(2\bar{V}_0y^4-\mu^2\right)}dy$.

Lie point symmetries of the WDW eq. can be used in order to construct Noether point symmetries for a Lagrangian of field equations (see arXiv:1312.3942). We have the following results.

- If $V_1 = 0$, then $a(t) = a_0\sqrt{t}$, i.e. the radiation solution
- If $V_1 \neq 0$, then $a(\tau) = a_0(\tau - \tau_0) + a_1 - a_2\frac{1}{\tau - \tau_0}$, where $dt = a(\tau)\,d\tau$. However, if $a_0 = 0$ the Friedmann eq. H^2 can be written

$$\frac{H^2}{H_0^2} = \Omega_{0,r}a^{-4} + \Omega_{0,m}a^{-3} + \Omega_{0,k}a^{-2} + \Omega_{0,f}a^{-1} + \Omega_{0,\Lambda}$$

where $\Omega_{0,i} = \Omega_{0,i}(a_1)$, $i = \{r, m, k, f, \Lambda\}$ and $a_2 = \frac{(|a_1|+1)^2}{H_0}$.

Similarly, we can find an exact classical solution for the second potential.

Conclusion

- The Lie point symmetries of the WDW eq. in Hybrid Gravity were studied.
- The Lie invariants were used in order to find exact solution of the WDW and to solve analytically the modified field equations.
- It is of interest that in the case of the power law potential $V(\phi) = V_0 \left(\sqrt{\phi} + V_1\right)^4$ the Friedmann equation H^2 is a fourth order polynomial with non vanishing coefficients; that is, every power law term of $\sqrt{\phi}$ in the potential produces a corresponding fluid in the model.

$$\frac{H^2}{H_0^2} = \Omega_{0,r} a^{-4} + \Omega_{0,m} a^{-3} + \Omega_{0,k} a^{-2} + \Omega_{0,f} a^{-1} + \Omega_{0,\Lambda}$$

Thank you!