Spinning compact binaries in general relativity through higher post-Newtonian orders (Hamiltonian treatment)

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Outline

- Some History on Hamiltonian General Relativity
- Spin in Minkowski Space
- Spin and Gravitation
- Self-Gravitating Spinning Objects: Canonical
- Applications
- Some History on Spin in General Relativity

higher order spin dynamics in collaboration with:
Damour, Hartung, Hergt, Jaranowski, Steinhoff, Wang, Zeng

often: $c = 1\,,\; G = 1$

Some History on Hamiltonian General Relativity

Dirac 1958-1959

1978 Nelson/Teitelboim: Dirac field

2009 Barausse/Racine/Buonanno: spinning test particles in "Kerr"

Arnowitt/Deser/Misner 1959-1960

1961 Kimura: 1PN

1974 Ohta/Okamura/Kimura/Hiida: 2PN (in part)

1985 Damour/GS: 2PN; GS: 2.5PN

1997 Jaranowski/GS: 3.5PN

2001 Damour/Jaranowski/GS: 3PN

2009 Steinhoff/GS: self-gravitating spinning particles

2014 Damour/Jaranowski/GS: 4PN

Schwinger 1963

1963 Kibble: Dirac field

Refinements

DeWitt 1967; Regge/Teitelboim 1974

Spin in Minkowski Space

el. charged particle with spin (m, e, $\mathbf{S}$)

$$\frac{dp_\mu}{d\tau} = eF_{\mu\nu}u^\nu$$

$$\eta_{\mu\nu}\frac{dw^\nu}{d\tau} = g\frac{e}{2m}F_{\mu\nu}w^\nu + (g-2)\frac{e}{2m^2}F_{\lambda\nu}u^\lambda w^\nu p_\mu$$

$$u^\mu = \tfrac{dx^\mu}{d\tau}, \quad -u^\mu u^\mu \eta_{\mu\nu} = 1, \quad p_\mu = mu_\mu, \quad -p_\mu p_\nu \eta^{\mu\nu} = m^2$$

$$w^\mu w^\nu \eta_{\mu\nu} = m^2\mathbf{S}^2, \quad p_\mu w^\mu = 0$$

$$w^0 = \mathbf{P}\cdot\mathbf{J} = \mathbf{P}\cdot\hat{\mathbf{S}}$$

$$\mathbf{w} = H\mathbf{J} + \mathbf{P}\times\mathbf{K} = H\hat{\mathbf{S}} - \tfrac{1}{H+m}\mathbf{P}\times(\hat{\mathbf{S}}\times\mathbf{P})$$

$$w^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}P_\nu J_{\alpha\beta} \qquad J_{kl} = \epsilon_{klm}J_m \qquad J_{0i} = K_i$$

$$w^\mu w_\mu \equiv -(w^0)^2 + \mathbf{w}^2 = m^2\hat{\mathbf{S}}^2$$

canonical variables

total angular momentum: $\mathbf{J} = \hat{\mathbf{X}} \times \mathbf{P} + \hat{\mathbf{S}}$

Lorentz boost: $\mathbf{K} = -\mathrm{t}\mathbf{P} + H\hat{\mathbf{X}} - \frac{1}{H+m}\hat{\mathbf{S}} \times \mathbf{P}$

Hamiltonian: $H = \sqrt{m^2 + \mathbf{P}^2}$

center-of-energy: $\bar{\mathbf{X}} = \hat{\mathbf{X}} - \frac{1}{(H+m)H}\,\hat{\mathbf{S}} \times \mathbf{P}$

$$\mathbf{K} = -\mathrm{t}\mathbf{P} + H\bar{\mathbf{X}}$$

Poincaré algebra

$$\{P_i, \mathrm{H}\} = \{J_i, H\} = 0$$

$$\{J_i, P_j\} = \varepsilon_{ijk}\, P_k \;, \;\; \{J_i, J_j\} = \varepsilon_{ijk}\, J_k$$

$$\{J_i, G_j\} = \varepsilon_{ijk}\, G_k$$

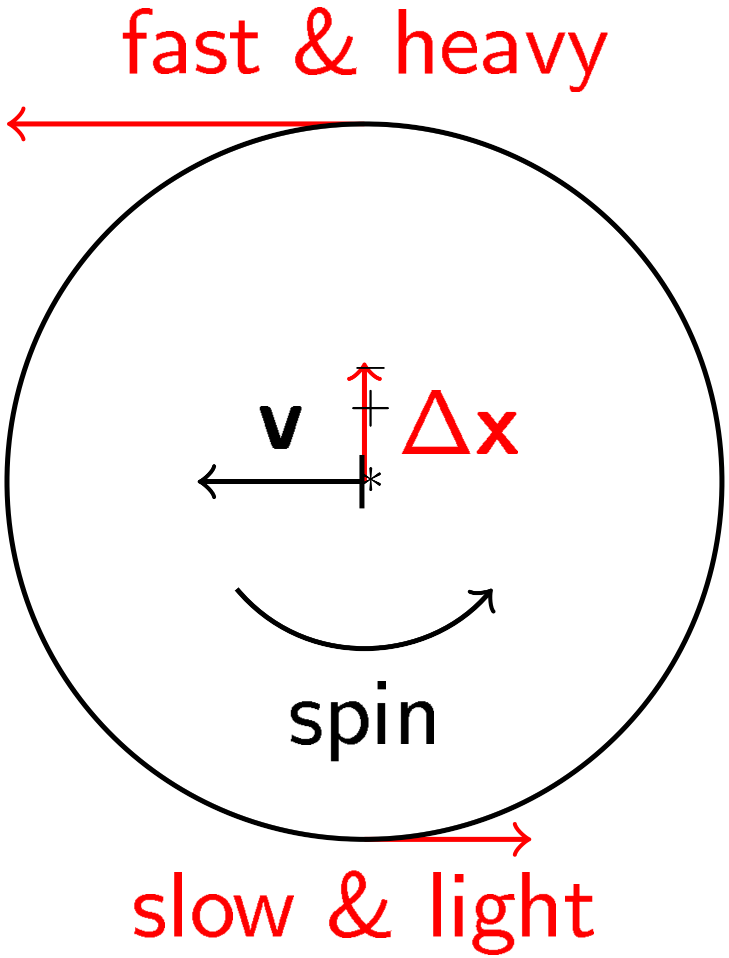
$$\{\mathbf{G}, H\} = \mathbf{P}$$

$$\{G_i, P_j\} = \frac{1}{c^2}\, H\, \delta_{ij}$$

$$\{G_i, G_j\} = -\frac{1}{c^2}\, \varepsilon_{ijk}\, J_k$$

Lorentz boost: $\quad \mathbf{K} = -t\,\mathbf{P} + \mathbf{G}$

$$d\mathbf{K}/dt = \partial\mathbf{K}/\partial t + \{\mathbf{K}, H\} = -\mathbf{P} + \{\mathbf{G}, H\} = 0$$



various centers: $\mathbf{X}(*)$, $\bar{\mathbf{X}}(-)$, $\hat{\mathbf{X}}(+)$

Steinhoff: Canonical Formulation of Spin in General Relativity,
Ann. Phys. (Berlin) 523, 296 (2011) [arXiv:1106.4203]

center-of-spin: $\hat{\mathbf{X}}$; $\{\hat{X}^i, \hat{X}^j\} = 0$ (Newton-Wigner coordinates)

center-of-energy: $\bar{\mathbf{X}} = \hat{\mathbf{X}} - \frac{1}{(H+m)H}\, \hat{\mathbf{S}} \times \mathbf{P}$

center-of-inertia: $\mathbf{X} = \hat{\mathbf{X}} + \frac{1}{(H+m)m}\, \hat{\mathbf{S}} \times \mathbf{P}$

center-of-inertia: $S^{\mu\nu} P_\nu = 0$

center-of-energy: $\bar{S}^{\mu\nu} n_\nu = 0, \qquad n_\mu = (-1, 0, 0, 0)$

center-of-spin: $m\hat{S}^{\mu\nu} n_\nu + \hat{S}^{\mu\nu} P_\nu = 0$

minimal radius of particle with spin: $\frac{\hat{S}}{mc}$

Compton wavelength: $2\frac{\hbar/2}{mc}$

radius of Kerr ring singularity (Boyer-Lindquist coordn.): $\frac{\hat{S}}{mc}$

Schwarzschild radius (Schwarzschild coordn.): $\frac{2Gm}{c^2}$

quantum massless particle

helicity: $\lambda\hbar = \frac{\mathbf{P}\cdot\hat{\mathbf{S}}}{H}$

$$\bar{\mathbf{X}} = \hat{\mathbf{X}} - \frac{1}{H^2}\,\hat{\mathbf{S}} \times \mathbf{P}$$

$$\mathbf{J} = \bar{\mathbf{X}} \times \mathbf{P} + \lambda\hbar\frac{\mathbf{P}}{H}$$

$$\bar{\mathbf{X}} \times \bar{\mathbf{X}} = i\lambda\hbar^2 \frac{\partial}{\partial \mathbf{P}} \frac{1}{|\mathbf{P}|}$$

$$\Delta\bar{X}_k \Delta\bar{X}_l \geq \frac{|\lambda|}{2}\hbar^2 | < P_m/H^3 > |, \qquad k \neq l \neq m$$

$$(\Delta\bar{\mathbf{X}})^2 \geq |\lambda|\hbar^2 < H^{-2} >$$

many particle systems with interaction

$$\mathbf{P} = \sum_a \mathbf{p}_a$$

$$\mathbf{J} = \sum_a (\mathbf{r}_a \times \mathbf{p}_a + \mathbf{s}_a)$$

$$M = \sqrt{H^2 - \mathbf{P}^2}, \qquad H = \sqrt{M^2 + \mathbf{P}^2}$$

$$\hat{\mathbf{X}} = \frac{\mathbf{G}}{H} + \frac{1}{M(H+M)}(\mathbf{J} - \frac{\mathbf{G}}{H} \times \mathbf{P}) \times \mathbf{P}$$

$$\{\hat{X}^i, \hat{X}^j\} = \{P^i, P^j\} = 0, \quad \{\hat{X}^i, P^j\} = \delta^{ij}$$

$$\{M, \hat{X}^j\} = \{M, P^j\} = \{M, H\} = 0$$

free particles with spin:

$$H = \sum_a h_a, \qquad h_a = \sqrt{m_a^2 + \mathbf{p}_a^2}$$

$$\mathbf{G} = \sum_a (h_a \mathbf{r}_a - \frac{1}{h_a + m_a} \mathbf{s}_a \times \mathbf{p}_a)$$

Schwinger: Particles, Sources and Fields I, 1998

Spin and Gravitation

tetrad field e_a^μ: $e_a^\mu e_{b\mu} = \eta_{ab}, \qquad e_{a\mu} e_{b\nu} \eta^{ab} = g_{\mu\nu} = g_{\nu\mu}$

local LT: $e'^\mu_a = L^b{}_a e_b^\mu, \qquad L^a{}_c \eta_{ab} L^b{}_d = \eta_{cd}$

linear connection ω_μ^{ab}: $D_\mu \phi \equiv \partial_\mu \phi + \frac{1}{2}\omega_\mu^{ab} G_{[ab]}\phi$

local LT: $\omega'^{ab}_\mu = L^a{}_c L^b{}_d \omega_\mu^{cd} + L^a{}_d \partial_\mu L^{bd}, \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$

inf. local LT: $\delta\phi = \delta\xi^{ab} G_{[ab]}\phi$

curvature tensor $R^{ab}{}_{\mu\nu}$: $D_\mu D_\nu \phi - D_\nu D_\mu \phi = R^{ab}{}_{\mu\nu} G_{[ab]}\phi$

$$R^{ab}{}_{\mu\nu} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\nu^{ac}\omega_\mu^{bd}\eta_{cd} - \omega_\mu^{ac}\omega_\nu^{bd}\eta_{cd}$$

Lagrangian for gravity

$$\mathcal{L}_G = \frac{1}{16\pi} \det(e^c_\gamma) e^\mu_a e^\nu_b R^{ab}{}_{\mu\nu}(\omega) + \partial_\mu \mathcal{C}^\mu$$

vacuum Einstein equations:

$$0 = \frac{\delta \mathcal{L}_G}{\delta e^\mu_a} e_{a\nu} \equiv 2\det(e^c_\gamma)(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)$$

$$0 = \frac{\delta \mathcal{L}_G}{\delta \omega^{ab}_\mu} \quad \Rightarrow \quad \omega^{ab}_\mu = \omega^{ab}_\mu(e, \partial_\nu e) \qquad \text{no torsion !}$$

Lagrangian for spinning objects

$$\mathcal{L}_M = \int d\tau \left[\left(p_\mu - \frac{1}{2} S_{ab}\, \omega_\mu{}^{ab} \right) \frac{dz^\mu}{d\tau} + \frac{1}{2} S_{ab} \frac{d\theta^{ab}}{d\tau} \right] \delta_{(4)}$$

$$\mathcal{L}_C = \int d\tau \left[\lambda_1^a p^b S_{ab} + \lambda_{2[i]} \Lambda^{[i]a} p_a - \frac{\lambda_3}{2} (p^2 + m^2) \right] \delta_{(4)}$$

$$d\theta^{ab} = \Lambda_C^{\ a} d\Lambda^{Cb} = -d\theta^{ba}$$

equations of motion

$$\frac{DS_{ab}}{D\tau} = 0$$

$$\frac{Dp_\mu}{D\tau} = -\frac{1}{2}\mathrm{R}^{(4)}_{\mu\rho ab} u^\rho S^{ab}$$

$$u^\mu \equiv \frac{dz^\mu}{d\tau} = \lambda_3 p^\mu$$

$$\sqrt{-g}T^{\mu\nu} = \int d\tau \left[\lambda_3 p^\mu p^\nu \delta_{(4)} + \left(u^{(\mu} S^{\nu)\alpha} \delta_{(4)}\right)_{||\alpha}\right]$$

Self-Gravitating Spinning Objects: Canonical

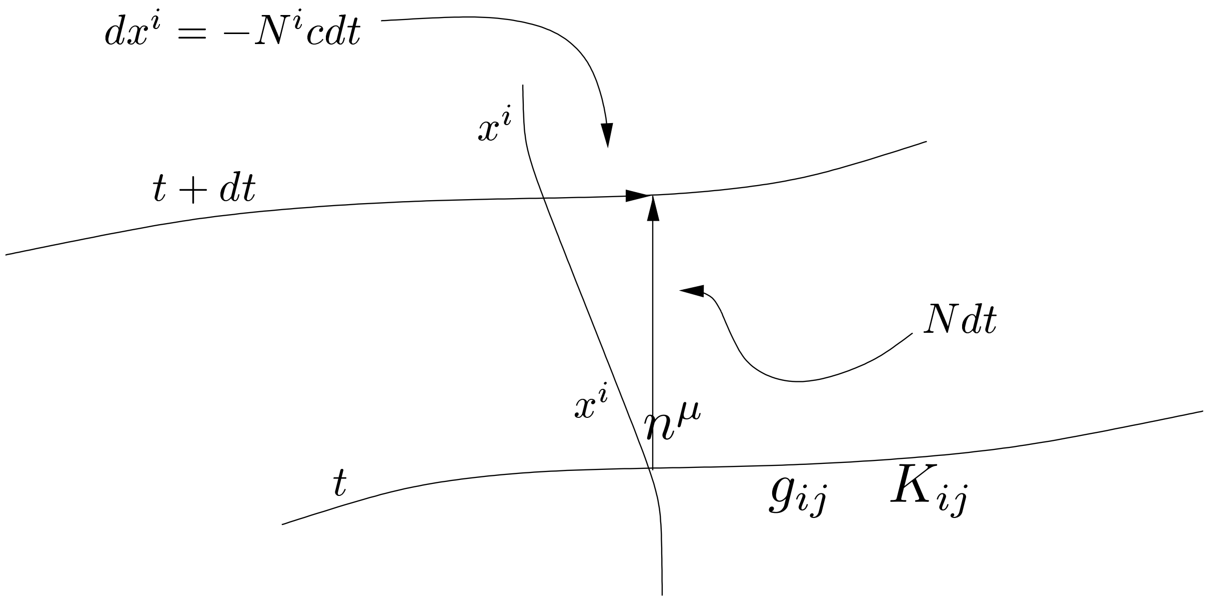
Steinhoff/GS: Europhys. Lett. 87:50004 (2009)

Steinhoff: Ann. Phys. (Berlin) 523, 296 (2011)

3+1 splitting of spacetime

$$n^\mu = (1, -N^i)/N$$

$$n_\mu = (-N, 0, 0, 0)$$



$$K_{ij} = -N\Gamma^0_{ij} = -Ng^{0\mu}(g_{i\mu,j} + g_{j\mu,i} - g_{ij,\mu})/2$$

$$\boxed{ds^2 = -(Ncdt)^2 + g_{ij}(dx^i + N^i cdt)(dx^j + N^j cdt)}$$

$$W = \frac{1}{16\pi}\int d^4x\, \pi^{ij}_{\text{can}} g_{ij,0} + \int dt \Big[P_i \dot{\hat{z}}^i + \frac{1}{2}\hat{S}_{(i)(j)}\dot{\hat{\theta}}^{(i)(j)} - H \Big]$$

$$H = \int d^3x (N\mathcal{H} - N^i\mathcal{H}_i) + \frac{c^4}{16\pi G}\oint_{i^0} d^2 s_i (g_{ij,j} - g_{jj,i})$$

$N|_{i^0} = 1 + \mathcal{O}(1/r), \quad N^i|_{i^0} = \mathcal{O}(1/r)$

If the constraints $\mathcal{H} \equiv \sqrt{\gamma}(T^{\mu\nu} - \frac{c^4}{8\pi G}G^{\mu\nu})n_\mu n_\nu = 0$ and $\mathcal{H}_i \equiv \sqrt{\gamma}(T^\mu_i - \frac{c^4}{8\pi G}G^\mu_i)n_\mu = 0$ are fulfilled and adapted coordinate conditions are applied, then

$$H = \frac{c^4}{16\pi G}\oint_{i^0} d^2 s_i (g_{ij,j} - g_{jj,i}) \equiv H_{\text{ADM}}[\pi^{ij\text{TT}}_{\text{can}}, h^{\text{TT}}_{ij}, P_i, \hat{z}^i, \hat{S}_{(i)(j)}]$$

$$W = \frac{1}{16\pi}\int d^4x\, \pi^{ij\text{TT}}_{\text{can}} h^{\text{TT}}_{ij,0} + \int dt \Big[P_i \dot{\hat{z}}^i + \frac{1}{2}\hat{S}_{(i)(j)}\dot{\hat{\theta}}^{(i)(j)} - H_{\text{ADM}} \Big]$$

solution of the matter constraints

$$n^\mu = (1, -N^i)/N, \qquad n_\mu = (-N, 0, 0, 0)$$

$$\lambda_3 : \quad np \equiv n^\mu p_\mu = -\sqrt{m^2 + \gamma^{ij} p_i p_j} \qquad\qquad \gamma^{ik} g_{kj} = \delta^i_j$$

$$\lambda_1 : \quad nS_i \equiv n^\mu S_{\mu i} = \frac{p_k \gamma^{kj} S_{ji}}{np} = g_{ij} n S^j$$

$$\lambda_2 : \quad \Lambda^{[j](0)} = \Lambda^{[j](i)} \frac{p_{(i)}}{p^{(0)}}, \quad \Lambda^{[0]a} = -\frac{p^a}{m}$$

time gauge for the tetrads

$$e^{\mu}_{(0)} = n^{\mu}, \qquad e^{0}_{(0)} = \frac{1}{N}, \qquad e^{i}_{(0)} = -\frac{N^i}{N}$$

$$g_{ij} = e_i^{(m)} e_{(m)j}$$

$$\mathcal{L}_{MC} = -N\mathcal{H}^{\mathrm{matter}} + N^i \mathcal{H}_i^{\mathrm{matter}}$$

$$\mathcal{H}^{\mathrm{matter}} = -np\delta - K^{ij}\frac{p_i n S_j}{np}\delta - (nS^k\delta)_{;k}$$

$$\mathcal{H}_i^{\mathrm{matter}} = (p_i + K_{ij} n S^j)\delta + \left(\frac{1}{2}\gamma^{mk} S_{ik}\delta + \delta_i^{(k}\gamma^{l)m}\frac{p_k n S_l}{np}\delta\right)_{;m}$$

transformation to canonical matter variables

$$z^i = \hat{z}^i - \frac{nS^i}{m - np}, \quad nS_i = -\frac{p_k \gamma^{kj} \hat{S}_{ji}}{m}$$

$$S_{ij} = \hat{S}_{ij} - \frac{p_i n S_j}{m - np} + \frac{p_j n S_i}{m - np}$$

$$\lambda^{[i](j)} = \hat{\lambda}^{[i](k)} \left(\delta_{kj} + \frac{p_{(k)} p^{(j)}}{m(m - np)} \right)$$

$$P_i = p_i + K_{ij} n S^j + \hat{A}^{kl} e_{(j)k} e^{(j)}_{l,i} - \left(\frac{1}{2} S_{kj} + \frac{p_{(k} n S_{j)}}{np} \right) \Gamma^{kj}{}_i$$

$$g_{ik} g_{jl} \hat{A}^{kl} = \frac{1}{2} \hat{S}_{ij} + \frac{m p_{(i} n S_{j)}}{np(m - np)}$$

$$S^{ab} S_{ab} = \hat{S}_{(i)(j)} \hat{S}_{(i)(j)} = 2 \hat{S}_{(i)} \hat{S}_{(i)} = 2 s^2 = \text{const}$$

$$\hat{\lambda}^{(i)}_{[k]} \hat{\lambda}^{[k](j)} = \delta_{ij}$$

$$d\hat{\theta}^{(i)(j)} \equiv \hat{\lambda}^{(i)}_{[k]} d\hat{\lambda}^{[k](j)} = -d\hat{\theta}^{(j)(i)}$$

adding Lagrangian of gravity

$$\hat{\mathcal{L}}_{MK} = P_i \dot{\hat{z}}^i \delta + \frac{1}{2} \hat{S}_{(i)(j)} \dot{\hat{\theta}}^{(i)(j)} \delta$$

$$\hat{\mathcal{L}}_{GK} = \hat{A}^{ij} e_{(k)i} e^{(k)}_{j,0} \delta$$

$$\hat{\mathcal{L}}_{GK} + \mathcal{L}_G = \frac{1}{8\pi} [\pi^{ij} + 8\pi \hat{A}^{ij} \delta] e_{(k)i} e^{(k)}_{j,0} \ + \ \mathcal{L}_{GC} - \frac{1}{16\pi} \mathcal{E}_{i,i}$$

$$\mathcal{E}_i = g_{ij,j} - g_{jj,i}$$

total energy: $E = \frac{1}{16\pi} \oint d^2 s_i \ \mathcal{E}_i = \frac{1}{16\pi} \int d^3 x \ \mathcal{E}_{i,i}$

$$\mathcal{L}_{GC} = -N\mathcal{H}^{\text{field}} + N^i\mathcal{H}_i^{\text{field}}$$

$$\mathcal{H}^{\text{field}} = -\frac{1}{16\pi\sqrt{\gamma}}\left[\gamma \mathrm{R} + \frac{1}{2}\left(g_{ij}\pi^{ij}\right)^2 - g_{ij}g_{kl}\pi^{ik}\pi^{jl}\right]$$

$$\mathcal{H}_i^{\text{field}} = \frac{1}{8\pi}g_{ij}\pi^{jk}{}_{;k}$$

$$\pi^{ij} = \sqrt{\gamma}(\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})K_{kl} \qquad \gamma \equiv \det(g_{ij})$$

spatially symmetric time gauge for the tetrads

$$e_{(k)i}e^{(k)}_{j,\mu} = B^{kl}_{ij} g_{kl,\mu} + \frac{1}{2} g_{ij,\mu}$$

$$e_{(i)j} = e_{ij} = e_{ji}$$

$$e_{ij}e_{jk} = g_{ik} \qquad e_{ij} = \sqrt{(g_{kl})}$$

$$2B^{ij}_{kl} = e_{mk}\frac{\partial e_{ml}}{\partial g_{ij}} - e_{ml}\frac{\partial e_{mk}}{\partial g_{ij}}$$

$$\pi^{ij}_{\mathrm{can}} = \pi^{ij} + 8\pi \hat{A}^{(ij)}\delta + 16\pi B^{ij}_{kl}\hat{A}^{[kl]}\delta$$

spacetime-coordinates conditions

$$3g_{ij,j} - g_{jj,i} = 0, \qquad \pi^{ii}_{\text{can}} = 0$$

$$g_{ij} = \Psi^4 \delta_{ij} + h^{\text{TT}}_{ij}, \qquad \pi^{ij}_{\text{can}} = \tilde{\pi}^{ij}_{\text{can}} + \pi^{ij\text{TT}}_{\text{can}}$$

transverse traceless: $h^{\text{TT}}_{ii} = \pi^{ii\text{TT}}_{\text{can}} = h^{\text{TT}}_{ij,j} = \pi^{ij\text{TT}}_{\text{can},j} = 0$

$$\tilde{\pi}^{ij}_{\text{can}} = V^i_{\text{can},j} + V^j_{\text{can},i} - \frac{2}{3}\delta_{ij} V^k_{\text{can},k}$$

constraints: $\mathcal{H}^{\text{field}} + \mathcal{H}^{\text{matter}} = 0, \qquad \mathcal{H}^{\text{field}}_i + \mathcal{H}^{\text{matter}}_i = 0$

total action in canonical form

$$W = \frac{1}{16\pi}\int d^4x\, \pi^{ij\mathrm{TT}}_{\mathrm{can}} h^{\mathrm{TT}}_{ij,0} + \int dt \left[P_i \dot{\hat{z}}^i + \frac{1}{2}\hat{S}_{(i)(j)}\dot{\hat{\theta}}^{(i)(j)} - E \right]$$

Hamiltonian: $E \equiv H_{\mathrm{ADM}} = -\frac{1}{2\pi}\int d^3x\; \Delta\Psi[\hat{z}^i, P_i, \hat{S}_{(i)(j)}, h^{\mathrm{TT}}_{ij}, \pi^{ij\mathrm{TT}}_{\mathrm{can}}]$

$$\{\hat{z}^i, P_j\} = \delta_{ij}, \qquad \{\hat{S}_{(i)}, \hat{S}_{(j)}\} = \epsilon_{ijk}\hat{S}_{(k)}$$

$$\{h^{\mathrm{TT}}_{ij}(\mathbf{x},t), \pi^{kl\mathrm{TT}}_{\mathrm{can}}(\mathbf{x}',t)\} = 16\pi\delta^{\mathrm{TT}kl}_{ij}\delta(\mathbf{x}-\mathbf{x}')$$

Spin-Gravity Hamiltonians (changed notations!)

leading order spin-orbit

$$H_{\rm SO}^{\rm LO} = \frac{G}{c^2} \sum_a \sum_{b \neq a} \frac{1}{r_{ab}^2} (\mathbf{S}_a \times \mathbf{n}_{ab}) \cdot \left[\frac{3m_b}{2m_a} \mathbf{p}_a - 2\mathbf{p}_b \right]$$

leading order spin(1)-spin(2)

$$H_{\rm S_1 S_2}^{\rm LO} = \frac{G}{c^2} \sum_a \sum_{b \neq a} \frac{1}{2r_{ab}^3} \left[3(\mathbf{S}_a \cdot \mathbf{n}_{ab})(\mathbf{S}_b \cdot \mathbf{n}_{ab}) - (\mathbf{S}_a \cdot \mathbf{S}_b) \right]$$

leading order spin(1)-spin(1)

$$H_{\rm S_1^2}^{\rm LO} = \frac{G}{c^2} \frac{m_2 C_{Q1}}{2m_1 r_{12}^3} \left[3(\mathbf{S}_1 \cdot \mathbf{n}_{12})(\mathbf{S}_1 \cdot \mathbf{n}_{12}) - (\mathbf{S}_1 \cdot \mathbf{S}_1) \right]$$

$$
\begin{aligned}
H_{\mathrm{SO}}^{\mathrm{NLO}} &= \frac{G}{c^4 r^2}\left[-\left((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12}\right)\left[\frac{5m_2\mathbf{p}_1^2}{8m_1^3}+\frac{3(\mathbf{p}_1\cdot\mathbf{p}_2)}{4m_1^2}\right.\right.\\
&- \left.\frac{3\mathbf{p}_2^2}{4m_1m_2}+\frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})}{4m_1^2}+\frac{3(\mathbf{p}_2\cdot\mathbf{n}_{12})^2}{2m_1m_2}\right]\\
&+ \left((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12}\right)\left[\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+\frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})}{m_1m_2}\right]\\
&+ \left.\left((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{p}_2\right)\left[\frac{2(\mathbf{p}_2\cdot\mathbf{n}_{12})}{m_1m_2}-\frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})}{4m_1^2}\right]\right]\\
&+ \frac{G^2}{c^4 r^3}\left[-\left((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12}\right)\left[\frac{11m_2}{2}+\frac{5m_2^2}{m_1}\right]\right.\\
&+ \left.\left((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12}\right)\left[6m_1+\frac{15m_2}{2}\right]\right]+(1\leftrightarrow 2)
\end{aligned}
$$

$$
\begin{aligned}
H^{\rm NLO}_{\rm S_1S_2} &= (G/2m_1m_2c^4r^3)[3((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_2\times\mathbf{S}_2)\cdot\mathbf{n}_{12})/2 \\
&+ 6((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) \\
&- 15(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
&- 3(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{p}_2) + 3(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) \\
&+ 3(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) + 3(\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
&+ 3(\mathbf{S}_2\cdot\mathbf{p}_2)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) - 3(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
&+ (\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{p}_2) - (\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{p}_1)/2 + (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)/2] \\
&+ (3/2m_1^2r^3)[-((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) \\
&+ (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{p}_1\cdot\mathbf{n}_{12})] \\
&+ (3/2m_2^2r^3)[-((\mathbf{p}_2\times\mathbf{S}_2)\cdot\mathbf{n}_{12})((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12}) \\
&+ (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})] \\
&+ (6G^2(m_1+m_2)/c^4r^4)[(\mathbf{S}_1\cdot\mathbf{S}_2) - 2(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})]
\end{aligned}
$$

$$
\begin{aligned}
H^{\mathrm{NLO}}_{\mathrm{S}_1^2} \quad = \quad & \frac{G}{c^4 r^3}\left[\frac{m_2}{m_1}\left(\left(-\frac{21}{8}+\frac{9}{4}C_{Q1}\right)\mathbf{p}_1^2\left(\mathbf{S}_1\cdot\mathbf{n}\right)^2\right.\right. \\
+ \quad & \left(\frac{15}{4}-\frac{9}{2}C_{Q1}\right)\left(\mathbf{p}_1\cdot\mathbf{n}\right)\left(\mathbf{S}_1\cdot\mathbf{n}\right)\left(\mathbf{p}_1\cdot\mathbf{S}_1\right) \\
+ \quad & \left(-\frac{5}{4}+\frac{3}{2}C_{Q1}\right)\left(\mathbf{p}_1\cdot\mathbf{S}_1\right)^2+\left(-\frac{9}{8}+\frac{3}{2}C_{Q1}\right)\left(\mathbf{p}_1\cdot\mathbf{n}\right)^2\mathbf{S}_1^2 \\
+ \quad & \left.\left(\frac{5}{4}-\frac{5}{4}C_{Q1}\right)\mathbf{p}_1^2\mathbf{S}_1^2\right)+\frac{1}{m_1^2}\left(-\frac{15}{4}C_{Q1}\left(\mathbf{p}_1\cdot\mathbf{n}\right)\left(\mathbf{p}_2\cdot\mathbf{n}\right)\left(\mathbf{S}_1\cdot\mathbf{n}\right)^2\right. \\
+ \quad & \left(3-\frac{21}{4}C_{Q1}\right)\left(\mathbf{p}_1\cdot\mathbf{p}_2\right)\left(\mathbf{S}_1\cdot\mathbf{n}\right)^2 \\
+ \quad & \left(-\frac{3}{2}+\frac{9}{2}C_{Q1}\right)\left(\mathbf{p}_2\cdot\mathbf{n}\right)\left(\mathbf{p}_1\cdot\mathbf{S}_1\right)\left(\mathbf{S}_1\cdot\mathbf{n}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \left(-3 + \frac{3}{2} C_{Q\,1} \right) (\mathbf{p}_1 \cdot \mathbf{n}) \, (\mathbf{p}_2 \cdot \mathbf{S}_1) \, (\mathbf{S}_1 \cdot \mathbf{n}) \\
&+ \quad \left(\frac{3}{2} - \frac{3}{2} C_{Q\,1} \right) (\mathbf{p}_1 \cdot \mathbf{S}_1) \, (\mathbf{p}_2 \cdot \mathbf{S}_1) + \left(\frac{3}{2} - \frac{3}{4} C_{Q\,1} \right) (\mathbf{p}_1 \cdot \mathbf{n}) \, (\mathbf{p}_2 \cdot \mathbf{n}) \, \mathbf{S}_1^2 \\
&+ \quad \left(-\frac{3}{2} + \frac{9}{4} C_{Q\,1} \right) (\mathbf{p}_1 \cdot \mathbf{p}_2) \, \mathbf{S}_1^2 \Bigg) + \frac{C_{Q\,1}}{m_1 m_2} \left(-\frac{3}{4} \mathbf{p}_2^2 \mathbf{S}_1^2 + \frac{9}{4} \mathbf{p}_2^2 \, (\mathbf{S}_1 \cdot \mathbf{n})^2 \right) \Bigg] \\
&+ \quad \frac{G^2 m_2}{c^4 r^4} \Bigg[\left(2 + \frac{1}{2} C_{Q\,1} + \frac{m_2}{m_1} (1 + 2 C_{Q\,1}) \right) \mathbf{S}_1^2 \\
&- \quad \left(3 + \frac{3}{2} C_{Q\,1} + \frac{m_2}{m_1} (1 + 6 C_{Q\,1}) \right) (\mathbf{S}_1 \cdot \mathbf{n})^2 \Bigg]
\end{aligned}
$$

black holes: $C_{Q\,1} = 1$, neutron stars: $C_{Q\,1} > 1$

$$\mathcal{H}^{\rm matter} = m_1\delta_1 + \mathcal{H}^{\rm matter}_{S_1^2, p_{1i}=0} + \frac{1}{2m_1}\mathbf{p}_1 \cdot (\mathbf{S}_1 \times \partial_1)\,\delta_1 + (1 \leftrightarrow 2)$$

$$\mathcal{H}^{\rm matter}_i = p_{1i}\delta_1 + \frac{1}{2}\left(\mathbf{S}_1 \times \partial_1\right)_i \delta_1 + (1 \leftrightarrow 2)$$

$$\mathcal{H}^{\rm matter}_{S_1^2, p_{1i}=0} = \Bigg[\left(\frac{1}{2}\gamma^{ki}\gamma^{lj}Q_{1ij}\delta_1\right)_{;kl} + \frac{1}{4m_1}\left(\gamma^{ij}\gamma^{mn}\gamma^{kl}{}_{,m}S_{1ln}S_{1jk}\delta_1\right)_{,i}$$

$$+ \frac{1}{8m_1}g_{mn}\gamma^{pj}\gamma^{ql}\gamma^{mi}{}_{,p}\gamma^{nk}{}_{,q}S_{1ij}S_{1kl}\delta_1\Bigg]$$

$$Q_{1ij} = \frac{C_{Q1}}{m_1}\left(\gamma^{kl}S_{1ik}S_{1jl} - \frac{1}{3}g_{ij}\gamma^{kl}\gamma^{mn}S_{1km}S_{1ln}\right)$$

SO

NLO: Tagoshi/Ohashi/Owen('01), Faye/Blanchet/Buonanno('06), Damour/Jaranowski/GS('08), Steinhoff/Hergt/GS('08), Levi('10), Porto('10)

NNLO: Hartung/Steinhoff('11), Marsat/Bohé/Faye/Blanchet('13)

DLO: Wang/Will('07), Steinhoff/Wang('10)

S1S2

NLO: Steinhoff/Hergt/GS('08), Porto/Rothstein('06,'08,'10), Levi('10)

NNLO: Hartung/Steinhoff('11), Levi('12)

DLO: Zeng/Will('07), Wang/Steinhoff/Zeng/GS('11)

S1S1

NLO[black holes]: Steinhoff/Hergt/GS('08), Porto/Rothstein('08,'10)

NLO[neutron stars]: Porto/Rothstein('08,'10), Steinhoff/Hergt/GS('10)

NLO center-of-mass:

$$\mathbf{G}^{\rm NLO}_{\rm SO} = -\sum_a \frac{\mathbf{P}_a^2}{8m_a^3}(\mathbf{P}_a \times \mathbf{S}_a)$$
$$+ \sum_a \sum_{b\neq a} \frac{m_b}{4m_a r_{ab}} \left[((\mathbf{P}_a \times \mathbf{S}_a)\cdot \mathbf{n}_{ab}) \frac{5\mathbf{x}_a + \mathbf{x}_b}{r_{ab}} - 5(\mathbf{P}_a \times \mathbf{S}_a) \right]$$
$$+ \sum_a \sum_{b\neq a} \frac{1}{r_{ab}} \left[\frac{3}{2}(\mathbf{P}_b \times \mathbf{S}_a) - \frac{1}{2}(\mathbf{n}_{ab} \times \mathbf{S}_a)(\mathbf{P}_b \cdot \mathbf{n}_{ab}) \right.$$
$$\left. - ((\mathbf{P}_a \times \mathbf{S}_a)\cdot \mathbf{n}_{ab}) \frac{\mathbf{x}_a + \mathbf{x}_b}{r_{ab}} \right]$$

$$\mathbf{G}^{\rm NLO}_{\rm S1S2} = \frac{1}{2} \sum_a \sum_{b\neq a} \left\{ [3(\mathbf{S}_a \cdot \mathbf{n}_{ab})(\mathbf{S}_b \cdot \mathbf{n}_{ab}) - (\mathbf{S}_a \cdot \mathbf{S}_b)] \frac{\mathbf{x}_a}{r_{ab}^3} + (\mathbf{S}_b \cdot \mathbf{n}_{ab}) \frac{\mathbf{S}_a}{r_{ab}^2} \right\}$$

Summary of known Hamiltonians

$$
\begin{aligned}
H_{\text{con}} &= H_N + H_{1PN} + H_{2PN} + H_{3PN} + H_{4PN} \\
&+ H_{SO}^{\text{LO}} + H_{S_1S_2}^{\text{LO}} + H_{S_1^2}^{\text{LO}} + H_{S_2^2}^{\text{LO}} \\
&+ H_{SO}^{\text{NLO}} + H_{S_1S_2}^{\text{NLO}} + H_{S_1^2}^{\text{NLO}} + H_{S_2^2}^{\text{NLO}} \\
&+ H_{SO}^{\text{NNLO}} + H_{S_1S_2}^{\text{NNLO}}
\end{aligned}
$$

$$
\begin{aligned}
H_{\text{diss}}(t) &= H_{2.5PN}(t) + H_{3.5PN}(t) \\
&+ H_{SO}^{\text{DLO}}(t) + H_{S_1S_2}^{\text{DLO}}(t)
\end{aligned}
$$

Applications

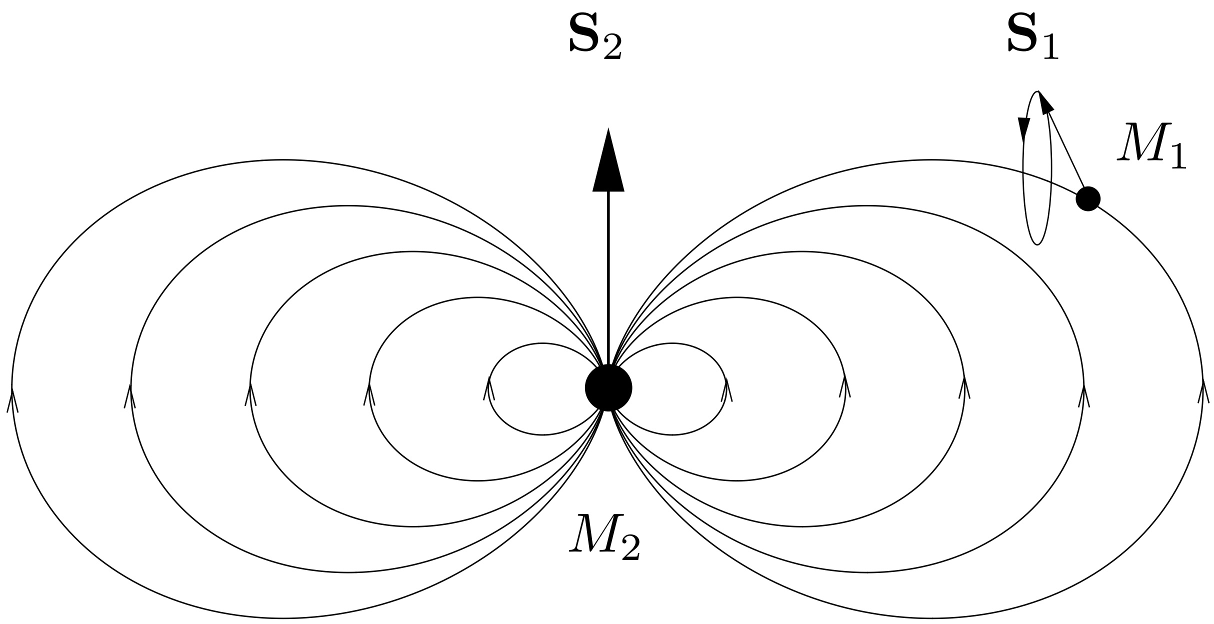
Leading-order spin-orbit coupling:

$$\boxed{H_{\rm SO} = \frac{2G}{c^2R^3}(\mathbf{S}\cdot\mathbf{L}) + \frac{3GM_1M_2}{2c^2R^3}(\mathbf{b}\cdot\mathbf{L})}$$

$$\mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2, \;\; \mathbf{b} \equiv \frac{\mathbf{S}_1}{M_1^2} + \frac{\mathbf{S}_2}{M_2^2}, \;\; \mathbf{L} \equiv \mathbf{R}\times\mathbf{P}, \;\; \mathbf{P} \equiv \mathbf{P}_1 = -\mathbf{P}_2$$

Leading-order spin-spin coupling:

$$\boxed{H_{\rm SS} = \frac{G}{2c^2R^3}\left(\frac{3(\mathbf{S}\cdot\mathbf{R})(\mathbf{S}\cdot\mathbf{R})}{R^2} - (\mathbf{S}\cdot\mathbf{S})\right)}$$


$\mathbf{S}_2$
$\mathbf{S}_1$
M_1
M_2

Lense-Thirring effect:

Runge-Lenz-Laplace vector: $\mathbf{A} \equiv \mathbf{P} \times \mathbf{L} - \frac{GM\mu^2}{R}\mathbf{R}$

$$< (\frac{d\mathbf{L}}{dt})^{\mathrm{SO}} >_t = \mathbf{\Omega}_{\mathrm{SO}} \times \mathbf{L}$$

$$< (\frac{d\mathbf{A}}{dt})^{\mathrm{SO}} >_t = \mathbf{\Omega}_{\mathrm{SO}} \times \mathbf{A}$$

$$\mathbf{\Omega}_{\mathrm{SO}} = \frac{2G}{c^2} < \frac{1}{R^3} >_t (\mathbf{S}_{\mathrm{eff}} - 3\frac{(\mathbf{L}\cdot\mathbf{S}_{\mathrm{eff}})\mathbf{L}}{L^2}), \qquad \mathbf{S}_{\mathrm{eff}} \equiv \mathbf{S} + \frac{3}{4}M_1 M_2 \mathbf{b}$$

LAGEOS(1&2): 31 mas/yr (1×10^{-1}, Ciufolini 2007)

Schiff effect (Lense-Thirring for spin, frame dragging):

$$(\frac{d\mathbf{S}_1}{dt})^{\mathrm{SS}} = \mathbf{\Omega}_{\mathrm{SS}} \times \mathbf{S}_1, \qquad \mathbf{\Omega}_{\mathrm{SS}} = \frac{G}{c^2 R^3}(3\frac{(\mathbf{R}\cdot\mathbf{S}_2)\mathbf{R}}{R^2} - \mathbf{S}_2)$$

GP-B: 39 mas/yr (2×10^{-1}, Everitt et al. 2011)

de Sitter effect (Fokker effect, geodetic precession):

$$(\frac{d\mathbf{S}_1}{dt})^{\mathrm{SO}} = \mathbf{\Omega}^{\mathrm{s}}_{\mathrm{SO}} \times \mathbf{S}_1, \qquad \mathbf{\Omega}^{\mathrm{s}}_{\mathrm{SO}} = \frac{2G}{c^2 R^3}(1 + \frac{3M_2}{4M_1})\mathbf{L}$$

$$\mathbf{L} = \frac{M_1 M_2}{M_1 + M_2}\mathbf{R} \times \mathbf{V}$$

Earth-Moon: 19 mas/yr (2×10^{-2}, Shapiro et al. 1988)

GP-B: 6,606 mas/yr (3×10^{-3}, Everitt et al. 2011)

The Binary Pulsar PSR B1913+16 (Hulse-Taylor Pulsar)
The Double Pulsar PSR J0737-3039B
Kramer: Relativistic Spin-Precession in Binary Pulsars [arXiv:1008.5032]

Some History on Spin in General Relativity

Quantum Spin (Electron)

1928 - 1929: Tetrode, Weyl, Fock

1932 - 1933: Schrödinger, Infeld/van der Waerden

1950: Weyl

1962 - 1963: Dirac, Kibble

1978: Nelson/Teitelboim

Classical Spin

1937: Mathisson

1951 - 1959: Papapetrou, Pirani, Tulczyjew

1964 - 1975: Taub, Dixon, Bailey/Israel

1975 - 1995: Barker/O'Connell, D'Eath, Thorne/Hartle, Kidder

2001 - 2011: as of above plus Barausse/Racine/Buonanno