BF Theory Explanation of the Entropy for Non-rotating Isolated Horizons

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The 1st Conference of Polish Society on Relativity
Spala, July. 1st, 2014

Based on the joint work with Jingbo Wang and Xu-an Zhao
PRD 89(2014) 084065

Outline

1. **Thermodynamics of BH and Isolated Horizon**
2. **Isolated Horizon Framework in LQG**

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2. **Isolated Horizon Framework in LQG**
3. **BF Theory Description of Isolated Horizon**
4. **Entropy of Isolated Horizon from BF Theory and LQG**

Thermodynamics of BH

Law	Thermodynamics	Black holes
Zeroth	T constant throughout body in thermal equilibrium	κ constant over horizon of a stationary black hole
First	$dU = TdS +$ work terms	$dM = \frac{\kappa}{8\pi}da + \Omega_H dJ + \Phi_H dQ$
Second	$\Delta S \geq 0$ in any process	$\Delta a \geq 0$ in any process
Third	Impossible to achieve $T = 0$ by a physical process	Impossible to achieve $\kappa = 0$ by a physical process

Figure: Engle and Liko, arXiv:11124412.

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- The three pillars of fundamental physics is brought together by

$$S_{BH} = \frac{k_B c^3 Ar_{BH}}{4G\hbar}.$$

Limitation of the global notions in GR

- The event horizon definition of BH requires knowledge of the entire space-time all the way to future null infinity.
- The use of stationary space-times to derive black hole thermodynamics is not ideal.
- The global nature of event horizon makes it difficult to use in quantum theory. In order for a definition of the horizon of black hole to make sense, one needs to be able to formulate it in terms of phase space functions which can be quantized.
- The global notions of ADM energy and ADM angular momentum are of limited use, because they do not distinguish the mass of black holes from the energy of surrounding gravitational radiation.

Quasi-local notion of Isolated Horizon

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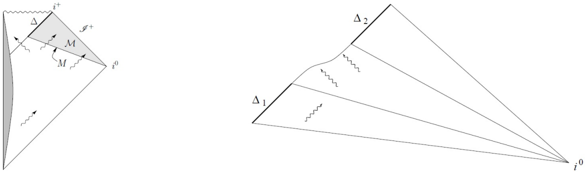
- (Weakly) Isolated Horizon: A three-dimensional null hypersurface Δ of a space-time (M, g_{ab}) is said to be a weakly isolated horizon if the following conditions hold:
  (1). Δ is topologically $\mathbb{R} \times S$ with S a compact two-dimensional manifold;
  (2). The expansion $\theta_{(l)}$ of any null normal l to Δ vanishes;
  (3). The field equations hold at Δ, and the stress-energy tensor T_{ab} of external matter fields is such that, at Δ, $-T^a{}_b l^b$ is a future-directed and causal vector for any future-directed null normal l^a.
  (4). An equivalence class $[l]$ of future-directed null normals is equipped with Δ, with $l' \sim l$ if $l' = cl$ ($c > 0$ a constant), such that $\mathcal{L}_l \omega_a \triangleq 0$ for all $l \in [l]$, where ω_a is related to the induced derivative operator D_a on Δ by $D_a l_b \triangleq \omega_a l_b$.

Quasi-local notion of Isolated Horizon



(a) (b)

Figure 1: (a) A typical gravitational collapse. The portion Δ of the horizon at late times is isolated. The space-time $\mathcal{M}$ of interest is the triangular region bounded by Δ, $\mathscr{I}^+$ and a partial Cauchy slice M. (b) Space-time diagram of a black hole which is initially in equilibrium, absorbs a small amount of radiation, and again settles down to equilibrium. Portions Δ_1 and Δ_2 of the horizon are isolated.

Figure: Ashtekar, Beetle and Fairhurst, gr-qc/9812065.

Thermodynamics of Isolated Horizon

- The definition of weakly isolated horizon implies automatically the zeroth law of IH mechanics as the surface gravity $\kappa_{(l)} \equiv \omega_a l^a$ is constant on Δ [Ashtekar, Beetle and Fairhurst, 1998].

Thermodynamics of Isolated Horizon

- The definition of weakly isolated horizon implies automatically the zeroth law of IH mechanics as the surface gravity $\kappa_{(l)} \equiv \omega_a l^a$ is constant on Δ [Ashtekar, Beetle and Fairhurst, 1998].
- Let us consider an 4-dimensional spacetime region $\mathcal{M}$ with an isolated horizon Δ as an inner boundary.
  The Hamiltonian framework for $\mathcal{M}$ provides an elegant way to define the quasi-local notions of energy E_Δ and angular momentum J_Δ associated to Δ.
- Then the first law of IH mechanics holds as [Ashtekar, Beetle and Lewandowski, 2001]

$$\delta E_\Delta = \frac{\kappa_{(l)}}{8\pi G}\delta a_\Delta + \Phi_{(l)}\delta Q_\Delta + \Omega_{(l)}\delta J_\Delta .$$

Basic variables in time gauge

Consider the Palatini action of GR on $\mathcal{M}$:

$$S[e,A] = -\frac{1}{4\kappa}\int_{\mathcal{M}} \varepsilon_{IJKL} e^I \wedge e^J \wedge F(A)^{KL} + \frac{1}{4\kappa}\int_{\tau_\infty} \varepsilon_{IJKL} e^I \wedge e^J \wedge A^{KL},$$

where $\kappa \equiv 8\pi G$.

- For later convenience, we define the solder form $\Sigma^{IJ} \equiv e^I \wedge e^J$ and its dual $(*\Sigma)_{KL} = \frac{1}{2}\varepsilon_{IJKL}\Sigma^{IJ}$.

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- For later convenience, we define the solder form $\Sigma^{IJ} \equiv e^I \wedge e^J$ and its dual $(*\Sigma)_{KL} = \frac{1}{2}\varepsilon_{IJKL}\Sigma^{IJ}$.
- Let the $so(3,1)$ connection A^{IJ} and the cotetrad e^I be in the time-gauge in which e_0^a is normal to the partial Cauchy surface M, reducing the internal local gauge group from $SO(1,3)$ to $SO(3)$.
- The pull-back of the spacetime variables to M can be written in terms of the Ashtekar-Barbero variables as

$$\mathcal{A}^i = \gamma A^{0i} - 1/2\epsilon^i{}_{jk}A^{jk}; \qquad \Sigma^i = \epsilon^i{}_{jk}\Sigma^{jk}.$$

Symplectic structure in time gauge

By using the covariant phase space approach [Lee and Wald, 1991], the symplectic structure of GR can be obtained on M with the inner boundary $H = M \cap \Delta$ as [Engle el, 2009]

$$\Omega(\delta_1, \delta_2) = \frac{1}{2\kappa\gamma} \int_M 2\delta_{[1}\Sigma^i \wedge \delta_{2]}\mathcal{A}_i - \frac{1}{\kappa} \frac{a_0}{\pi(1-\gamma^2)} \oint_H 2\delta_{[1}\mathcal{A}_i \wedge \delta_{2]}\mathcal{A}^i .$$

- The symplectic structure consists of a bulk term, the standard symplectic structure used in LQG, and a surface term, the symplectic structure of an $SU(2)$ Chern-Simons theory on H.

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- The symplectic structure consists of a bulk term, the standard symplectic structure used in LQG, and a surface term, the symplectic structure of an $SU(2)$ Chern-Simons theory on H.
- In deriving the above symplectic structure, one makes key use of the fact that, in terms of the Ashtekar-Barbero variables, the isolated horizon boundary conditions take the form

$$\Sigma^i = -\frac{a_0}{\pi(1-\gamma^2)} F^i(\mathcal{A}).$$

Kinematical structure of LQG

- In canonical LQG, the kinematical Hilbert space is spanned by spin network states $|\Gamma, \{j_e\}, \{i_v\} >$, where Γ denotes some graph in the spatial manifold M, each edge e of Γ is labeled by a half-integer j_e and each vertex v is labeled by an intertwiner i_v.

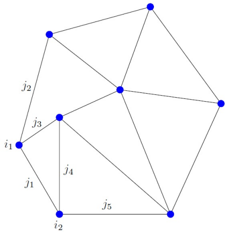

Figure: Dona and Speziale, arXiv:1007.0402.

Quantum isolated horizon

- In the case when M has a boundary H, some edges of Γ may intersect H and endow it a quantum area at each puncture.

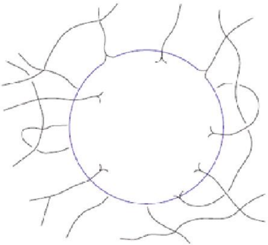

Figure: Ashtekar, Baez and Krasnov, gr-qc/0005126.

Calculation of the entropy for IH

- To calculate the entropy for IH, one constructs the micro-canonical ensemble by considering only the subspace of the bulk theory with a fixed area of the horizon.
- Employing the spectrum of the area operator in LQG, a detail analysis can estimates the number of Chern-Simons surface states consistent with the given area [Ashtekar, Baez and Krasnov, 2000]. One thus obtains the IH entropy, whose leading term is indeed proportional to the horizon area.

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- However, the expression of the entropy agrees with the Hawking-Bekenstein formula only if one chooses a particular Barbero-Immirzi parameter $\gamma \approx 0.274$ [Domagala and Lewandowski, 2004; Agullo el, 2009].
- The sub-leading term of the entropy has also been calculated and shown to be proportional to the logarithm of the horizon area [Das, Kaul and Majumdar, 2000].

Non-rotating IH

- Limitation: The above framework is only valid for even-dimensional spacetime, since Chern-Simons theory can only lives on odd-dimensional manifold.
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  Is there any way out?
- To describe the geometry near the isolated horizon Δ, it is convenient to employ the Newman-Ponrose formalism with the null tetrad $(l, n, m, \bar{m})$ adapted to Δ, such that the real vectors l and n coincide with the outgoing and ingoing future directed null vectors at Δ respectively.
- The gravitational contribution to angular momentum of the horizon is coded in the imaginary part of the Weyl tensor component $\Psi_2 = C_{abcd} l^a m^b \bar{m}^c n^d$ [Ashtekar, Beetle and Lewandowski, 2001].
- We will consider non-rotating isolated horizons satisfying $Im(\Psi_2) \triangleq 0$.

The symplectic flux across the IH

- The second-order variation of the previous Palatini action on $\mathcal{M}$ leads to the conservation identity of the symplectic current as

$$\frac{1}{\kappa}\Big(\int_{M_1}\delta_{[1}(*\Sigma)_{IJ}\wedge\delta_{2]}A^{IJ}-\int_{M_2}\delta_{[1}(*\Sigma)_{IJ}\wedge\delta_{2]}A^{IJ} + \int_{\Delta}\delta_{[1}(*\Sigma)_{IJ}\wedge\delta_{2]}A^{IJ}\Big)=0,$$

where M_1, M_2 are spacelike boundary of $\mathcal{M}$.

- We can show that the symplectic flux across the horizon can be expressed as a sum of two terms corresponding to the two-sphere $H_1=\Delta\cap M_1$ and $H_2=\Delta\cap M_2$.

Near horizon coordinates

- In the neighborhood of Δ, we choose the Bondi-like coordinates given by (v, r, x^i), $i = 1, 2$, where the horizon is given by $r = 0$ [Lewandowski, 2000]..

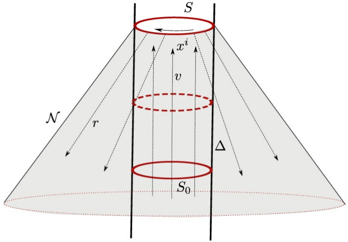

Figure 11: The near horizon coordinates. The isolated horizon is Δ and the transverse null surface is $\mathcal{N}$. The affine parameter along the outgoing null geodesics on $\mathcal{N}$ is r, and v is a coordinate along the null generators on Δ, and x^1 are coordinates on the cross-sections of Δ.

Figure: Krishnan, arXiv:1303.4635.

Gauge choice of the tetrad

- We choose an appropriate set of co-tetrad fields which are compatible with the metric as:

$$e^0 = \sqrt{\frac{1}{2}}(\alpha n + \frac{1}{\alpha} l),\ e^1 = \sqrt{\frac{1}{2}}(\alpha n - \frac{1}{\alpha} l),$$
$$e^2 = \sqrt{\frac{1}{2}}(m + \bar{m}),\ e^3 = i\sqrt{\frac{1}{2}}(m - \bar{m}),$$

  where $\alpha(x)$ is an arbitrary function of the coordinates.
- Each choice of $\alpha(x)$ characterizes a local Lorentz frame in the plane $\mathcal{I}$ formed by $\{e^0, e^1\}$.

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  where $\alpha(x)$ is an arbitrary function of the coordinates.
- Each choice of $\alpha(x)$ characterizes a local Lorentz frame in the plane $\mathcal{I}$ formed by $\{e^0, e^1\}$.
- Restricted to the horizon Δ, the co-tetrad fields read

$$e^0 \triangleq e^1 \triangleq \sqrt{1/2}\alpha n,\ \ e^2 \triangleq \sqrt{2} Re(\xi_i^{(0)}) dx^i,\ e^3 \triangleq -\sqrt{2} Im(\xi_i^{(0)}) dx^i.$$

  where $\xi_i^{(0)}$ are only functions of (x^1, x^2) satisfying $\xi_i^{(0)}\xi_{(0)}^i = 0$ and $\xi_i^{(0)}\bar{\xi}_{(0)}^i = 1$.

Variables restricted to the horizon

- The non-vanishing solder fields Σ^{IJ} restricted to Δ satisfy:

$$\Sigma^{0i} \triangleq \Sigma^{1i}, \forall i = 2, 3,$$
$$\Sigma^{23} = i\bar{m} \wedge m \triangleq -2Im(\xi_1^{(0)}\bar{\xi}_2^{(0)})dx^1 \wedge dx^2.$$

- We can also get the following properties for the connection restricted to Δ:

$$A^{0i} \triangleq A^{1i}, \forall i = 2, 3,$$
$$A^{01} \triangleq d\beta(x),$$

where $\beta(x) = \tilde{\kappa}v + \ln\alpha(x)$.

Horizon degrees of freedom

- The horizon integral of the symplectic current can be reduced to

$$\frac{1}{\kappa}\int_{\Delta}\delta_{[1}(*\Sigma)_{IJ}\wedge\delta_{2]}A^{IJ}=\frac{2}{\kappa}\int_{\Delta}\delta_{[1}\Sigma^{23}\wedge\delta_{2]}A^{01}.$$

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$$\frac{1}{\kappa}\int_\Delta \delta_{[1}(*\Sigma)_{IJ}\wedge\delta_{2]}A^{IJ} = \frac{2}{\kappa}\int_\Delta \delta_{[1}\Sigma^{23}\wedge\delta_{2]}A^{01}.$$
- Since the property of isolated horizon ensures that the area of the slice is unchanged for different v, we have
$$d(*\Sigma)_{01} = d\Sigma^{23} \triangleq 0.$$
Thus Σ^{23} is closed.
- So we can define an 1-form $\tilde{B}$ locally on Δ such that
$$\Sigma^{23} = d\tilde{B}. \tag{1}$$

Horizon gauge freedom and symplectic structure

- We have the following condition for the integral over any cross section of Δ:

$$\oint_{S^2} d\tilde{B} = \oint_{S^2} \Sigma^{23} = -\oint_{S^2} 2Im(\xi_1^{(0)}\bar{\xi}_2^{(0)})dx^1 \wedge dx^2 = a_H.$$

where a_H represents the area of the horizon.

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  where a_H represents the area of the horizon.

- Under a SO(1,1) boost on the plane spanned by $\{e^0, e^1\}$ with group element $g = exp(\zeta)$, we get $A^{'01} = A^{01} - d\zeta$ and $\Sigma'_{23} = \Sigma_{23}$ unchanged.
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- The full symplectic structure can be obtained as

$$\Omega(\delta_1, \delta_2) = \frac{1}{2\kappa\gamma}\int_M 2\delta_{[1}\Sigma^i \wedge \delta_{2]}\mathcal{A}_i + \frac{1}{\kappa}\oint_H 2\delta_{[1}\Sigma^1 \wedge \delta_{2]}\beta.$$

3D $SO(1,1)$ BF theory

- In 3-dimensional space-time Σ without boundary, the action of the SO(1,1) BF theory can be written as

$$S[B, A] = \int_{\Sigma} B \wedge F(A) = \int_{\Sigma} dB \wedge A.$$

- The equations of motion easily read

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- In the Hamiltonian formalism, the restriction of the field A and B to the spatial hypersurface satisfy the Gaussian constraint as well as the constraint: $F = dA = 0$.
  The latter generates gauge transformations of the form:

$$A \to A, \quad B \to B + d\lambda.$$

Symplectic structure of the *BF* theory

- The symplectic form on the covariant phase space of the *BF* theory reads

$$\Omega(\delta_1, \delta_2) = \oint_{\tilde{H}} 2\delta_{[2} B \wedge \delta_{1]} A.$$

- If $\tilde{H}$ is topologically a two-sphere and the B field is not globally defined, the above integration can still be well defined as the sum of integrals over two topological trivial patches and one of their boundaries.

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- The symplectic form on the covariant phase space of the *BF* theory reads

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- If $\tilde{H}$ is topologically a two-sphere and the B field is not globally defined, the above integration can still be well defined as the sum of integrals over two topological trivial patches and one of their boundaries.
- It turns out that the previous horizon boundary symplectic form of the Palatini theory can be regarded as that of SO(1,1) BF theory by making the identification:

$$B \leftrightarrow \frac{\tilde{B}}{\kappa},\ A \leftrightarrow A^{01}.$$

Quantum *BF* theory with sources

- To adapt the structure of LQG in the bulk, the boundary *BF* theory is "punctured" with sources as

$$F = dA = 0, \quad dB = \frac{\Sigma^1}{2\kappa},$$

- Since the bulk field Σ^1 is the source of the B field rather than the A field, after quantization the so-called so(1,1) connection A could not feel the "punctures" where the quantum geometric excitations locate.

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- Let's assume that the graph Γ underling a spin network state intersects H by n punctures denoted by $\mathcal{P} = \{p_i | i = 1, \cdots, n\}$.
  For every puncture p_i we associated a bounded neighborhood s_i which contains it and does not intersect with each other. We denote the boundary of s_i by η_i .

Quantum *BF* theory with sources

- The physical degrees of freedom of our "punctured" BF theory are encoded in the flux functions

$$f_i = \int_{s_i} dB = \oint_{\eta_i} B,$$

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- The configuration space of the BF theory with n "punctures" is $\mathbb{R}^n$.

  we can employ the well-known Lebesque measure to define the quantum Hilbert space $\mathcal{H}_H^{\mathcal{P}}$ as the space of L^2 functions on $\mathbb{R}^n$.

- There is a spectral decomposition of $\mathcal{H}_H^{\mathcal{P}}$ with respect to each configuration operator $\hat{f}_i$, i.e,

$$(\{a_p\}, \mathcal{P}|\hat{f}_i = (\{a_p\}, \mathcal{P}|a_i.$$

Quantum horizon boundary condition

- Consider the bulk kinematical Hilbert space $\mathcal{H}_M^{\mathcal{P}}$ defined on a graph $\Gamma \subset M$ with $\mathcal{P}$ as the set of its end points on H. $\mathcal{H}_M^{\mathcal{P}}$ can be spanned by the spin network states $|\mathcal{P}, \{j_p, m_p\}; \cdots >$, where j_p and m_p are respectively the spin labels and magnetic numbers of the edge e_p with $p \in \mathcal{P}$.

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- The integral $\Sigma^1(H) = \int_H \Sigma^1$ can be promoted as an operator:

$$\hat{\Sigma}^1(H)|\mathcal{P}, \{j_p, m_p\}; \cdots >= 16\pi\gamma l_{Pl}^2 \sum_{p \in \Gamma \cap H} m_p |\mathcal{P}, \{j_p, m_p\}; \cdots > .$$

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- The equations of the boundary BF theory motive us to input the quantum version of the horizon boundary condition as

$$(Id \otimes \hat{f}_i(s_i) - \frac{\hat{\Sigma}^1(s_i)}{2\kappa} \otimes Id)(\Psi_v \otimes \Psi_b) = 0,$$

where $\Psi_v \in \mathcal{H}_M^{\mathcal{P}}$ and $\Psi_b \in \mathcal{H}_H^{\mathcal{P}}$.

Solving the quantum boundary condition

- The space of kinematical states on a fixed Γ, satisfying the boundary condition, can be written as

$$\mathcal{H}_\Gamma = \bigoplus_{\{j_p, m_p\}_{p\in\Gamma\cap H}} \mathcal{H}_M^{\mathcal{P}}(\{j_p, m_p\}) \otimes \mathcal{H}_H^{\mathcal{P}}(\{m_p\}),$$

where $\mathcal{H}_H^{\mathcal{P}}(\{m_p\})$ denotes the subspace corresponds to the spectrum $\{m_p\}$ in the spectral decomposition of $\mathcal{H}_H^{\mathcal{P}}$ with respect to the operators $\hat{f}_p$ on the boundary.

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- The imposition of the diffeomorphism constraint implies that one only needs to consider the diffeomorphism equivalence class of quantum states.
  Hence, in the following states counting, we will only take account of the number of "punctures" on H, while the possible position of "punctures" are irrelevant.

Area constraint

- For the bulk Hilbert space $\mathcal{H}_M^{\mathcal{P}}$ with a horizon boundary H, the flux-area operator $\hat{a}_H^{flux}$ corresponding to the classical area $\int_H |dB|$ of H can also be naturally defined as [Barbero, Lewandowski and Villasenor, 2009]

$$\hat{a}_H^{flux}|\mathcal{P}, \{j_p, m_p\}; \cdots >= 8\pi\gamma l_{Pl}^2 (\sum_{p=1}^{n} |m_p|)|\mathcal{P}, \{j_p, m_p\}; \cdots > .$$

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- We have the area constraint:

$$\sum_{p\in\mathcal{P}} |m_p| = a, \quad m_p \in \mathbb{N}/2,$$

where $a = \dfrac{a_H}{8\pi\gamma l_{Pl}^2}$.

States counting

- For a given horizon area a_H, the horizon states satisfying the boundary condition are labeled by sequences $(m_1, \cdots, m_n)$ subject to area constraint, where $2m_i$ are integers.
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- The dimension of the horizon Hilbert space compatible with the given macroscopic horizon area can be calculated as:

$$\mathcal{N} = \sum_{n=0}^{n=2a-1} C_{2a-1}^{n} 2^{n+1} = 2 \times 3^{2a-1},$$

where C_i^j are the binomial coefficients.

Entropy of IH

- The entropy for an arbitrary non-rotating isolated horizon is given by

$$S = \ln \mathcal{N} = (2 \ln 3) a + \ln \frac{2}{3} = \frac{\ln 3}{\pi \gamma} \frac{a_H}{4 l_{Pl}^2} + \ln \frac{2}{3}.$$

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- If we fix the value of the Barbero-Immirzi parameter as $\gamma = \frac{\ln 3}{\pi}$, which is different from its value predicted in the Chern-Simons approach, the Bekenstein-Hawking area law can be obtained.
- The quantum correction to the Bekenstein-Hawking area law in our approach is a constant $\ln(2/3)$ rather than a logarithmic term.

Remarks

- In the Chern-Simons theory description of the horizon, the boundary degrees of freedom are encoded in the Chern-Simons connection.
  However, in our BF theory description, the connection becomes pure gauge, while the non-trivial degrees of freedom of the horizon are all encoded in the B field.

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- Our value for the Barbero-Immirzi parameter coincides with its value obtained in a particular case in [Barbero, Lewandowski and Villasenor, 2009] by employing the same flux-area operator as ours but still in the approach of Chern-Simons theory.
- Taking account of the fact that LQG can be extended to arbitrary spacetime dimensions[Bodendorfer, Thiemann and Thurn, 2011], the virtue of our BF theory approach is that it admits extension to arbitrary dimensional horizons.

!Thanks!