Fedosov quantization and geometric non-commutative gravity

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Fedosov product of endomorphisms

- On symplectic manifold $(\mathcal{M}, \omega)$ with symplectic connection ∂^S (torsion-free, $\partial^S \omega = 0$) there exist canonical coordinate covariant Fedosov $*$-product of functions.
- Let $\mathcal{E}$ be a vector bundle over $\mathcal{M}$. Let $\mathrm{End}(\mathcal{E})$ be corresponding bundle of endomorphisms.
- Sections of $\mathrm{End}(\mathcal{E})$ can be multiplied in the natural manner. Locally this is just usual matrix multiplication.
- Fedosov product can be generalized to sections of $\mathrm{End}(\mathcal{E})$.
- For this purpose we need also some connection $\partial^{\mathcal{E}}$ in $\mathcal{E}$.

Fedosov product of endomorphisms

$$A * B = AB - \frac{\mathrm{i}h}{2}\omega^{ab}\partial_a A \partial_b B - \frac{h^2}{8}\omega^{ab}\omega^{cd}\Big(\{\partial_b A, R^{\mathcal{E}}_{ac}\}\partial_d B + \partial_b A\{R^{\mathcal{E}}_{ac}, \partial_d B\} + \partial_{(a}\partial_{c)} A \partial_{(b}\partial_{d)} B\Big) + O(h^3).$$

$*$-product isomorphisms

- Having two symplectic connections ∂_1^S, $\partial_1^{\mathcal{E}}$ and two connections in the vector bundle ∂_2^S, $\partial_2^{\mathcal{E}}$ one can build two $*$-products: $*_1$ and $*_2$.
- They are equivalent, i.e. there exists isomorphism $M(f *_1 g) = M(f) *_2 M(g)$.
- Such isomorphisms can be quite easily constructed and enumerated.

Trace functional

- There is only one (up to normalizing constant) family of functionals fulfilling the requirements
  - $\mathrm{tr}_*(A * B) = \mathrm{tr}_*(B * A)$
  - $\mathrm{tr}_{*_1}(F) = \mathrm{tr}_{*_2}(M(F))$ where M is arbitrary $*$-isomorphism between $*_1$ and $*_2$.

Trace functional

$$\mathrm{tr}_*(A) = \int_{\mathcal{M}} \mathrm{Tr}\left(A + \frac{\mathrm{i}h}{2}\omega^{ab} R^{\mathcal{E}}_{ab} A \right.$$
$$\left. + h^2\left(-\frac{3}{8}\omega^{[ab}\omega^{cd]} R^{\mathcal{E}}_{ab} R^{\mathcal{E}}_{cd} + s_2\right) A + O(h^3)\right)\frac{\omega^n}{n!},$$

where

$$s_2 = \frac{1}{64}\omega^{[ab}\omega^{cd]} \overset{S}{R}{}^k{}_{lab} \overset{S}{R}{}^l{}_{kcd} + \frac{1}{48}\omega^{ab}\omega^{cd} \overset{S}{\partial}_e \overset{S}{\partial}_a \overset{S}{R}{}^e{}_{bcd},$$

and Tr stands for the trace of a matrix.

Seiberg-Witten map as $*$-isomorphism

Start with Fedosov $*$-product of endomorphisms generated by the connection $\partial^{\mathcal{E}}$.

- Choose some frame e in $\mathcal{E}$.
- Construct $*$-isomorphism $M_{\langle e \rangle}$ between initial $*$-product and $*_S$.
- Now, let us switch to a different frame $\tilde{e} = eg^{-1}$. In the analogous way we can construct $*$-isomorphism $M_{\langle \tilde{e} \rangle}$.

Seiberg-Witten map as $*$-isomorphism

- What is the relation between $M_{\langle e\rangle}$ and $M_{\langle\tilde{e}\rangle}$?
- It turns out that

$$M_{\langle\tilde{e}\rangle}(B_{\langle\tilde{e}\rangle}) = \widehat{g}_{\langle e\rangle}\left(g,\Gamma^{\mathcal{E}}\right) *_S M_{\langle e\rangle}(B_{\langle e\rangle}) *_S \widehat{g}^{-1}_{\langle e\rangle}\left(g,\Gamma^{\mathcal{E}}\right)$$

- Above formula is nothing but the deformation of the usual covariance relation $B_{\langle\tilde{e}\rangle} = gB_{\langle e\rangle}g^{-1}$ in the Seiberg-Witten-like style.
- Seiberg-Witten map appears as a local *consequence* of global Fedosov quantization.

General scheme

1. Take some Fedosov manifold $\mathcal{M}$ and an action, which leads to vacuum general relativity.
2. Rewrite the action by representing Lagrangian as a trace of (product of) endomorphism of some bundle.
3. (Replace the product of endomorphisms by Fedosov $*$-product of endomorphisms).
4. Replace the integral by Fedosov trace functional.
5. Calculate the variations to obtain field equations.
6. Observe that steps *3* and *4* induce that the theory is locally equivalent to the theory with Seiberg-Witten map applied on endomorphisms.

Incompatibility of volume forms

- In tr_* the symplectic $\mathrm{vol}_S = \frac{\omega^n}{n!}$ volume form is used, while in GR actions the metric one $\mathrm{vol}_M = \sqrt{-g}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{2n}$ appears.
- Define $v : \mathcal{M} \to \mathbb{R}$ by $\mathrm{vol}_M = v\,\mathrm{vol}_S$.
- We can fix proper (from GR point of view) volume form by rescaling one of endomorphisms by v. Let $\breve{A} := vA$.

Ricci tensor as an endomorphism of $T\mathcal{M}$

- Consider Einstein-Hilbert action $\mathcal{S}_{EH} = \int_{\mathcal{M}} R \,\mathrm{vol}_M$.
- Rewrite it as

$$\mathcal{S}_{EH_{1A}} = \int_{\mathcal{M}} \mathrm{Tr}\, \breve{\underline{R}}\, \frac{\omega^n}{n!},$$

  where $\underline{R}$ denotes endomorphism of $T\mathcal{M}$ given by Ricci tensor, i.e. $(\underline{R}X)^i = R^i{}_j X^j$.
- Fedosov construction requires connection in $T\mathcal{M}$. Take ∇ – Levi-Civita connection of underlying metric.

Ricci tensor as an endomorphism of $T\mathcal{M}$

After deformation, the action reads

$$\widehat{\mathcal{S}}_{EH_{1A}} = \mathrm{tr}_{*_{EH_1}}(\underline{\breve{R}}) = \\ = \int_{\mathcal{M}} \left(R - \frac{3}{8} h^2 X^{k\ l}_{\ \ l\ m} R^{m}{}_{k} + h^2 s_2 R + O(h^3) \right) \mathrm{vol}_M ,$$

where

$$X^{ijkl} := \omega^{[ab}\omega^{cd]} R^{ij}{}_{ab} R^{kl}{}_{cd} .$$

Ricci tensor as an endomorphism of $T\mathcal{M}$

From variation of the metric one obtains field equations

$$R^{ab} - \frac{1}{2}g^{ab}R + h^2\Bigg[\frac{3}{8}\Bigg(- R^{(a}{}_k X_l{}^{b)kl} + \frac{1}{2}R^k{}_l X^l{}_m{}^m{}_k g^{ab} + \nabla_k \nabla^{(a} X^{b)}{}_l{}^{lk}$$
$$- \frac{1}{2}\nabla_l \nabla^l X^a{}_k{}^{kb} - \frac{1}{2}g^{ab}\nabla_k \nabla_l X^k{}_m{}^{ml} - 2\nabla_k \nabla^l \left(R^{(a}{}_m Y_l{}^{b)mk}\right)$$
$$+ 2\nabla_k \nabla_l \left(R^{km} Y^{l(a}{}_m{}^{b)}\right)\Bigg) - \frac{1}{2}g^{ab}Rs_2 + R^{ab}s_2 + g^{ab}\nabla_l \nabla^l s_2$$
$$- \nabla^a \nabla^b s_2\Bigg] + O(h^3) = 0,$$

for

$$Y^{ijkl} := \omega^{[ij}\omega^{ab]}R^{kl}{}_{ab}.$$

Ricci tensor as an endomorphism of $T\mathcal{M}$

Write a metric as a formal series $g_{ab} = \overset{(0)}{g}_{ab} + h\overset{(1)}{g}_{ab} + h^2\overset{(2)}{g}_{ab} + \dots$ and put it into field equations.

- $\overset{(0)}{g}_{ab}$ is just classical Ricci-flat metric.
- $\overset{(1)}{g}_{ab}$ is just classical first order perturbation of $\overset{(0)}{g}_{ab}$
- for $\overset{(1)}{g}_{ab} = 0$ (no classical first order perturbation)

$$\overset{(2)}{g}_{ab} = -\frac{3}{8}\overset{(0)}{X}_{ak}{}^{k}{}_{b} - \frac{1}{n-1}\left(s_2 - \frac{3}{16}\overset{(0)}{X}_{mk}{}^{km}\right)\overset{(0)}{g}_{ab}$$

with $\overset{(0)}{X}{}^{ijkl} = \omega^{[ab}\omega^{cd]}\overset{(0)}{R}{}^{ij}{}_{ab}\overset{(0)}{R}{}^{kl}{}_{cd}$ and

$$s_2 = \frac{1}{64}\omega^{[ab}\omega^{cd]}\overset{S}{R}{}^{k}{}_{lab}\overset{S}{R}{}^{l}{}_{kcd} + \frac{1}{48}\omega^{ab}\omega^{cd}\overset{S}{\partial}_{e}\overset{S}{\partial}_{a}\overset{S}{R}{}^{e}{}_{bcd}$$

Interpretation in terms of SW map

- Non-commutativity of spacetime emerges in local interpretation by means of Seiberg-Witten map.
- For the case of flat ∂^S the following local formula hold

$$\widehat{\mathcal{S}}_{EH_{1A}} = \mathrm{tr}_{*_{EH_1}}(\underline{\breve{R}}) = \int_{\mathbb{R}^{2n}} \mathrm{Tr}(\widehat{\underline{\breve{R}}})\, \mathrm{d}^{2n}x$$

- Such local version of action is invariant under noncommutative gauge transformations $\widehat{L} \to \widehat{g} *_M \widehat{L} *_M \widehat{g}^{-1}$

How can we go further?

- One can construct theories with dynamical non-commutativity.
- Some generalization of Fedosov theory to put metric tensor "inside" it.

(A bit) generalized Fedosov theory

Main object of Fedosov construction is Weyl algebras bundle.

Its fibre consist of "functions" (formal power series) defined on fibre of tangent bundle

$$a(y) = \sum_{k,p \geq 0} h^k a_{i_1 \dots i_p} y^{i_1} \dots y^{i_p} \quad y \in T_x \mathcal{M}$$

The fiberwise product is is defined by the Moyal formula

$$a \circ b = \sum_{m=0}^{\infty} \left(-\frac{\mathrm{i}h}{2} \right)^m \frac{1}{m!} \frac{\partial^m a}{\partial y^{i_1} \dots \partial y^{i_m}} \omega^{i_1 j_1} \dots \omega^{i_m j_m} \frac{\partial^m b}{\partial y^{j_1} \dots \partial y^{j_m}}$$

(A bit) generalized Fedosov theory

The idea is to consider different fiberwise product

$$a \overset{\sim}{\circ} b = g^{-1}(ga \circ gb)$$

where

$$g = \mathrm{id} + \sum_{\substack{2s-k\geq 0 \\ s,k>0}} h^s g_{(s)}^{i_1 \ldots i_k} \frac{\partial^k}{\partial y^{i_1} \ldots \partial y^{i_k}}$$

(A bit) generalized Fedosov theory

For example with

$$g = \exp\left(\frac{\mathrm{i}h}{4} g^{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j}\right)$$

one obtains

$$a \overset{\sim}{\circ} b = \sum_{m=0}^{\infty} \left(-\frac{\mathrm{i}h}{2}\right)^m \frac{1}{m!} \frac{\partial^m a}{\partial y^{i_1} \dots \partial y^{i_m}} s^{i_1 j_1} \dots s^{i_m j_m} \frac{\partial^m b}{\partial y^{j_1} \dots \partial y^{j_m}}$$

where $s^{ij} = \omega^{ij} + g^{ij}$.

(A bit) generalized Fedosov theory

- Current status: Fedosov construction works fine for generalized fiberwise product ($*$-product, isomorphism theory, trace functional)
- Explicit formulae – ?

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Seiberg-Witten equations from Fedosov deformation quantization of endomorphism bundle

Int. J. Geom. Meth. Mod. Phys. **8** (2011), 411, [arXiv:0904.4409]

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Some models of geometric noncommutative general relativity

Phys. Rev. D **84** (2011), 065005 [arXiv:1011.0165]

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Involution in quantized endomorphism bundle and reality of noncommutative gravity actions

Int. J. Geom. Meth. Mod. Phys. **10** (2013), 1220029 [arXiv:1202.3287]