Twisting and κ-Poincaré

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- One of the approaches to the theory describing Planck scale is to consider noncommutative spacetimes and quantum deformed symmetries.
- Deformed Poincare (Hopf) algebra plays role of deformed relativistic symmetry for such noncommutative spacetime.
- One of the types of noncommutative spacetime is when coordinates satisfy the particular Lie algebra type commutation relations. It is called κ-Minkowski spacetime and is covariant under the κ-deformed Poincare algebra as deformed relativistic symmetry.

1. cocycle twist $F \in H \otimes H$ fulfils the 2-cocycle and normalization conditions:

$$F_{12}(\Delta \otimes id)(F) = F_{23}(id \otimes \Delta)F \ , \quad (\epsilon \otimes id)(F) = 1 = (id \otimes \epsilon)(F)$$

- provides standard deformation of classical Hopf algebras leading to the coproducts: $\Delta^F = F\Delta_0 F^{-1}$

2. cochain twists - leads in general case to quasi-Hopf algebra with universal R-matrix $\mathcal{R}$: $\Delta^{op}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \quad , \quad a \in H$; with non-unital coassociator $\phi \in H \otimes H \otimes H$ $(\phi \neq 1 \otimes 1 \otimes 1)$ modifying the quasi-triangularity relations for the universal R-matrix as follows:

$$\begin{aligned}(\Delta \otimes id)(\mathcal{R}) &= \phi_{312}\mathcal{R}_{13}\phi^{-1}_{132}\mathcal{R}_{23}\phi_{123}\\ (id \otimes \Delta)(\mathcal{R}) &= \phi^{-1}_{231}\mathcal{R}_{13}\phi_{213}\mathcal{R}_{12}\phi^{-1}_{123}\end{aligned}$$

For the cocycle twists : $\phi = 1 \otimes 1 \otimes 1$

- Until now the universal R-matrix for κ-Poincaré Hopf algebra is not known in D=2 and D=4.
- The description of quantum deformation by twist provides explicit formula for universal R-matrix.
- It is commonly accepted that the 2-cocycle twist providing κ-Poincaré Hopf algebra should not exist.
- We will show that the coproducts of quantum κ-Poincaré algebra (in the classical algebra basis) also can not be obtained by the cochain ($\phi \neq 1 \otimes 1 \otimes 1$) twist depending only on Poincaré algebra generators.

- Let's assume that κ-Poincaré-Hopf algebra $U_\kappa(\hat{g})$ ($\hat{g} = (P_\mu, M_{\mu\nu})$) can be obtained from cochain twist: $F \in U_\kappa(\hat{g}) \otimes U_\kappa(\hat{g})$ and that κ-deformed coproducts are of the form: $\Delta_\kappa = F\Delta_0 F^{-1}$
- We can expand the twist into the power series in $\frac{1}{\kappa}$ as follows

$$F = \exp\left(\frac{1}{\kappa}f_1 + \frac{1}{\kappa^2}f_2 + O\left(\frac{1}{\kappa^3}\right)\right) \tag{1}$$

- This implies the following perturbative formula for the deformed coproducts $\Delta \in U_\kappa(\hat{g}) \otimes U_\kappa(\hat{g})$

$$\Delta^F = F\Delta_0 F^{-1} = \Delta_0 + \frac{1}{\kappa}\Delta_1 + \frac{1}{\kappa^2}\Delta_2 + O\left(\frac{1}{\kappa^3}\right) =$$

$$= \Delta_0 + \frac{1}{\kappa}[f_1, \Delta_0] + \frac{1}{2\kappa^2}[f_1, [f_1, \Delta_0]] + \frac{1}{\kappa^2}[f_2, \Delta_0] + O\left(\frac{1}{\kappa^3}\right) \tag{2}$$

I. D=2 $\kappa-$Poincaré algebra in bicrossproduct basis is described by two-momentum generators $P_\mu = (P_0, P_1)$ and boost generator N with properties:

i) algebra

$$[\mathcal{P}_0, \mathcal{P}_1] = 0 \quad , \quad [N, \mathcal{P}_0] = i\mathcal{P}_1$$
$$[N, \mathcal{P}_1] = \frac{i}{2}\kappa\left(1 - \exp\left(-\frac{2\mathcal{P}_0}{\kappa}\right)\right) + \frac{i}{2\kappa}\mathcal{P}_1^2$$

ii) coalgebra

$$\Delta(\mathcal{P}_0) = \mathcal{P}_0 \otimes 1 + 1 \otimes \mathcal{P}_0 \quad , \quad \Delta(\mathcal{P}_1) = \mathcal{P}_1 \otimes 1 + \exp\left(-\frac{\mathcal{P}_0}{\kappa}\right) \otimes \mathcal{P}_1$$
$$\Delta(N) = N \otimes 1 + \exp\left(-\frac{\mathcal{P}_0}{\kappa}\right) \otimes N$$

We can derive D=2 quantum $\kappa-$Poincaré Hopf algebra in classical basis from the following inverse quantum map

$$P_0 = \frac{\kappa}{2}\left(\exp\left(\frac{\mathcal{P}_0}{\kappa}\right) - \exp\left(-\frac{\mathcal{P}_0}{\kappa}\right)(1 - \frac{1}{\kappa^2}P_1^2)\right) \quad , \quad P_1 = \mathcal{P}_1 \exp\left(\frac{\mathcal{P}_0}{\kappa}\right)$$

D=2 quantum $\kappa-$Poincaré Hopf algebra in classical basis

$$[P_0, P_1] = 0 \quad , \quad [N, P_0] = iP_1 \quad , \quad [N, P_1] = iP_0$$

$$\begin{aligned} \Delta(P_0) &= P_0 \otimes \Pi_0 + \Pi_0^{-1} \otimes P_0 + \frac{1}{\kappa} P_1 \Pi_0^{-1} \otimes P_1 \\ \Delta(P_1) &= P_1 \otimes \Pi_0 + 1 \otimes P_1 \\ \Delta(N) &= N \otimes 1 + \Pi_0^{-1} \otimes N \end{aligned}$$

where

$$\Pi_0 = \frac{1}{\kappa} P_0 + \sqrt{1 - \frac{1}{\kappa^2} C_0} \quad , \quad \Pi_0^{-1} = \frac{\sqrt{1 - \frac{1}{\kappa^2} C_0} - \frac{1}{\kappa} P_0}{1 - \frac{1}{\kappa^2} P_1^2}$$

with C_0 describing the standard undeformed mass Casimir

$$C_0 = P^0 P_0 + P^1 P_1 = -P_0^2 + P_1^2$$

and $\kappa-$deformed mass Casimir

$$C = \kappa^2 \left(\Pi_0 + \Pi_0^{-1} - 2 + \frac{1}{\kappa^2} P_1^2 \Pi_0^{-1} \right)$$

We expand the coproducts in classical basis in powers of $\frac{1}{\kappa}$ using

$$\Pi_0 = 1+\frac{1}{\kappa}P_0-\frac{1}{2\kappa^2}C_0+O\left(\frac{1}{\kappa^3}\right) \quad , \quad \Pi_0^{-1} = 1-\frac{1}{\kappa}P_0+\frac{1}{\kappa^2}\left(P_0^2+\frac{1}{2}C_0\right)+O\left(\frac{1}{\kappa^3}\right)$$

and get

$$\Delta_\kappa(P_0) = P_0\otimes 1+1\otimes P_0+\frac{1}{\kappa}P_1\otimes P_1+$$

$$+\frac{1}{\kappa^2}\left(P_0^2\otimes P_0+\frac{1}{2}C_0\otimes P_0-\frac{1}{2}P_0\otimes C_0-P_1P_0\otimes P_1\right)+O\left(\frac{1}{\kappa^3}\right);$$

$$\Delta_\kappa(P_1) = P_1\otimes 1+1\otimes P_1+\frac{1}{\kappa}P_1\otimes P_0-\frac{1}{2\kappa^2}P_1\otimes C_0+O\left(\frac{1}{\kappa^3}\right);$$

$$\Delta_\kappa(N) = N\otimes 1+1\otimes N-\frac{1}{\kappa}P_0\otimes N+\frac{1}{\kappa^2}\left(P_0^2\otimes N+\frac{1}{2}C_0\otimes N\right)+O\left(\frac{1}{\kappa^3}\right)$$

- From $\Delta_1 = [f_1, \Delta_0]$ and $\Delta_1(N) = -P_0 \otimes N$ one can easily calculate that:
$$f_1 = -iP_1 \otimes N \tag{3}$$
i.e. we get 'half' of classical r-matrix because $r = f_1 - f_1^T$.
- The equation determining the term f_2 looks as follows:
$$\Delta_2 = \frac{1}{2}[f_1, [f_1, \Delta_0]] + [f_2, \Delta_0] \tag{4}$$
where Δ_2 are given explicitly by formulas from expanded coproducts.
- We shall show that such f_2 which should provide $\frac{1}{\kappa^2}$ terms in the coproducts does not exists.

- Let us notice that due to $f_1 = -iP_1 \otimes N$ we get $[f_1, [f_1, \Delta_0 (N)]] = 0$.
- From $\Delta_2 (N) = P_0^2 \otimes N + \frac{1}{2} C_0 \otimes N$ we see that the left factors of tensor product are quadratic in P and the right ones the terms linear in N.
- Such property due to $\Delta_2 = \frac{1}{2} [f_1, [f_1, \Delta_0]] + [f_2, \Delta_0]$ implies that if $f_2 = A_\alpha \otimes B_\alpha$, the factors A_α have to be quadratic in momenta and factors B_α linear in N.
- In such circumstances the most general ansatz for f_2 is the following:

$$f_2 = \alpha P_0^2 \otimes N + \beta P_1^2 \otimes N + \gamma P_0 P_1 \otimes N + f_2^{(0)} \tag{5}$$

where $[f_2^{(0)}, \Delta_0(N)] = 0$.

- Using such f_2 we get

$$\begin{aligned}\Delta_2(N) &= \alpha\left[P_0^2,N\right]\otimes N+\beta\left[P_1^2,N\right]\otimes N+\gamma\left[P_0P_1,N\right]\otimes N= \\ &= -i\left(\left(2\alpha+2\beta\right)P_0P_1+\gamma\left(P_0P_0+P_1P_1\right)\right)\otimes N\end{aligned}$$

- Comparing this result with $\Delta_2(N)=\frac{1}{2}(P_0^2+P_1^2)\otimes N$ we obtain that:

$$-i\gamma=\frac{1}{2};\alpha+\beta=0$$

- This implies that

$$f_2=\frac{i}{2}P_0P_1\otimes N+\beta C_0\otimes N+f_2^{(0)}$$

It is easy to see that for $f_2 = \frac{i}{2} P_0 P_1 \otimes N + \beta C_0 \otimes N + f_2^{(0)}$ the terms

- $P_0 \otimes C_0$ in

$$\Delta_2 (P_0) = P_0^2 \otimes P_0 + \frac{1}{2} C_0 \otimes P_0 - \frac{1}{2} P_0 \otimes C_0 - P_1 P_0 \otimes P_1$$

- and $P_1 \otimes C_0$ in

$$\Delta_2 (P_1) = -\frac{1}{2} P_1 \otimes C_0$$

cannot be obtained (for any $f_2^{(0)}$) from the formula

$$\Delta_2 = \frac{1}{2} [f_1, [f_1, \Delta_0]] + [f_2, \Delta_0]$$

In particular, the term $P_0 \otimes P_0^2$ can never be obtained from the commutator $[f_2, P_0 \otimes 1 + 1 \otimes P_0]$ for any choice of f_2.
A similar argument one can use in $D = 4$ case.

II. κ-Poincaré from twisting - D=4

$$
\begin{aligned}
[M_i, M_j] &= i\epsilon_{ijk} M_k \quad , \quad [M_i, N_j] = i\epsilon_{ijk} N_k \quad , \quad [N_i, N_j] = -i\epsilon_{ijk} M_k \\
[M_j, P_k] &= i\epsilon_{jki} P_i \quad , \quad [M_j, P_0] = 0 \quad , \quad [N_j, P_0] = iP_j \quad , \quad [N_i, P_j] = i\delta_{ij} P_0
\end{aligned}
$$

$$
\begin{aligned}
\Delta(P_0) &= P_0 \otimes \Pi_0 + \Pi_0^{-1} \otimes P_0 + \frac{1}{\kappa} P_k \Pi_0^{-1} \otimes P_k \\
\Delta(P_k) &= P_k \otimes \Pi_0 + 1 \otimes P_k \\
\Delta(M_i) &= M_i \otimes 1 + 1 \otimes M_i \\
\Delta(N_i) &= N_i \otimes 1 + \Pi_0^{-1} \otimes N_i - \frac{1}{\kappa} \epsilon_{ikj} P_k \Pi_0^{-1} \otimes M_j
\end{aligned}
$$

where

$$
\Pi_0 = \frac{1}{\kappa} P_0 + \sqrt{1 - \frac{1}{\kappa^2} C_0} \quad , \quad \Pi_0^{-1} = \frac{\sqrt{1 - \frac{1}{\kappa^2} C_0} - \frac{1}{\kappa} P_0}{1 - \frac{1}{\kappa^2} P_1^2}
$$

We expand coproducts (Π_0 and Π_0^{-1}) in $\frac{1}{\kappa}$ power series as before, with four-dimensional classical mass Casimir C_0.

Expanding these coproducts in $\frac{1}{\kappa}$ we get

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 + \frac{1}{\kappa} P_k \otimes P_k + \tag{6}$$

$$+\frac{1}{\kappa^2}\left(P_0^2 \otimes P_0 + \frac{1}{2} C_0 \otimes P_0 - P_k P_0 \otimes P_k - \frac{1}{2} P_0 \otimes C_0\right) + O\left(\frac{1}{\kappa^3}\right)$$

$$\Delta(P_k) = P_k \otimes 1 + 1 \otimes P_k + \frac{1}{\kappa} P_k \otimes P_0 - \frac{1}{2\kappa^2} P_k \otimes C_0 + O\left(\frac{1}{\kappa^3}\right) \tag{7}$$

$$\Delta(N_i) = N_i \otimes 1 + 1 \otimes N_i - \frac{1}{\kappa}\left(\epsilon_{ikj} P_k \otimes M_j + P_0 \otimes N_i\right) + \tag{8}$$

$$+\frac{1}{\kappa^2}\left(\left(P_0^2 + \frac{1}{2} C_0\right) \otimes N_i + \epsilon_{ikj} P_k P_0 \otimes M_j\right) + O\left(\frac{1}{\kappa^3}\right)$$

Following the form of f_1 in 2 dimensional case, one can postulate the following formula for D=4 $\kappa-$deformation

$$f_1 = -iP_i \otimes N_i \tag{9}$$

One can check that with such f_1 one gets correctly the linear terms in the above coproducts (6)-(8).

After using formulae $\Delta_2 = \frac{1}{2}[f_1,[f_1,\Delta_0]] + [f_2,\Delta_0]$ and $\Delta(P_0)$ we present the term $\Delta_2(P_0)$ in two ways:

$$\begin{aligned}
\Delta_2(P_0) &= -\frac{1}{2}[P_i \otimes N_i, [P_j \otimes N_j, P_0 \otimes 1 + 1 \otimes P_0]] + [f_2, P_0 \otimes 1 + 1 \otimes P_0] = \\
&= \frac{1}{2}\vec{P}^2 \otimes P_0 + [f_2, P_0 \otimes 1 + 1 \otimes P_0] \\
&\stackrel{?}{=} \frac{1}{2}\left(P_0 \otimes P_0^2 + P_0^2 \otimes P_0 + \vec{P}^2 \otimes P_0 - P_0 \otimes \vec{P}^2\right) - P_k P_0 \otimes P_k \qquad (10)
\end{aligned}$$

In analogy to the case D=2 we can show that it is impossible to find such $f_2 = A_\alpha \otimes B_\alpha$ that leads to the validity of last equality in (10) (we can not get from cochain twist the term $P_0 \otimes P_0^2$).

C. Final remarks

- Such cochain twist can be provided only if we enlarge the Poincaré symmetries, in particular by the scale transformations - identified with the dilatations generator $\mathcal{D}$. In such a way the twist F can be introduced as spanned by the generators of the eleven-dimensional extension $(P_\mu, M_{\mu\nu}, \mathcal{D})$ of the D=4 Poincaré algebra called also D=4 Weyl algebra.

- We did show as well the non-existence of cochain twist in the standard non-classical basis of κ-Poincaré algebra. It appears that our results are valid in arbitrary basis for κ-Poincaré.