One-parameter dependence of DeWitt's metric in Vilkoviski-DeWitt formalism

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$$\int \mathcal{D}h_\Lambda \exp\left\{\int \mathrm{d}^4x\, \mathscr{L}(\ell, h)\right\} = \exp\left\{-\int \mathrm{d}^4x\, \mathscr{L}_{\mathit{eff}}(\ell, \Lambda)\right\},$$

gdzie

$$\mathscr{L}(\ell, h) = \mathscr{L}_0(h) + \mathscr{L}_0(\ell) + \mathscr{L}_{\text{int}}(h, \ell) \quad \rightarrow \quad \mathscr{L}_{\mathit{eff}}(\ell, \Lambda) = \mathscr{L}_0(\ell) + \sum_{i\in\mathbb{N}} c_i(\Lambda)\mathcal{O}^i(\ell)\,.$$

- Low energy dynamics (i.e., below a fundamental scale) does not depend on the details of a high energy one. The latter is encoded in coupling constants that accompany local vertices;
- Depending on the desired accuracy only a finite number of terms is required;
- Above a fundamental scale it is should be sewn with a more fundamental theory;

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Gravity as an Effective Field Theory

$$\mathscr{L}(g) = \Lambda + \frac{1}{\kappa^2}R + a_1 R^2_{\mu\nu} + a_2 R^2 + a_3 M^{-2} R^3 + \dots$$

J. F. Donoghue, *General Relativity as an effective field theory: the leading quantum corrections*, Phys.Rev.D**50** (1994) 3874;

Geometry of Configuration Space

Consider the theory with a gauge symmetry generated by $K^i_\alpha(\varphi) \in \mathsf{G}$, i.e.

$$\delta_\xi S[\varphi] = S_{,i}[\varphi]\delta_\xi \varphi^i = 0,$$

where

$$\delta_\xi \varphi^i = K^i_\alpha(\varphi)\delta\xi^\alpha, \qquad K^i_\alpha(\varphi)K^j_{\beta,i}(\varphi) - K^i_\beta(\varphi)K^j_{\alpha,i}(\varphi) = f^\gamma_{\alpha\beta}(\varphi)K^j_\gamma(\varphi).$$

Let the configuration field space $\mathcal{F}$ be a manifold endowed with a metric $\gamma(\varphi)$. Due to the underlying gauge symmetry it follows that

$$\mathrm{d}\varphi^i = \mathrm{d}\varphi^i_\perp + \mathrm{d}\varphi^i_\parallel, \quad \mathrm{d}\varphi^i_\parallel = K^i_\alpha(\varphi)\mathrm{d}\varepsilon^\alpha \quad \text{and} \quad \mathrm{d}\varphi^i_\perp = P^i_j \mathrm{d}\varphi^i,$$

where P^i_j is a projector on the space orthogonal to the gauge transformations. The metric on $\mathcal{F}$ can be cast into the form

$$\mathrm{d}s^2 = \gamma_{ij}\mathrm{d}\varphi^i\mathrm{d}\varphi^j = \gamma^\perp_{ij}\mathrm{d}\varphi^i_\perp\mathrm{d}\varphi^j_\perp + N_{\alpha\beta}\mathrm{d}\varepsilon^\alpha\mathrm{d}\varepsilon^\beta,$$

where

$$N_{\alpha\beta} \equiv K^i_\alpha\gamma_{ij}K^j_\beta \quad \text{and} \quad N_{\alpha\beta}N^{\beta\gamma} = \delta^\gamma_\alpha, \quad \Rightarrow \quad P^i_j = \delta^i_j - K^i_\alpha N^{\alpha\beta}K^i_\beta\gamma_{ij},$$

The metric on the orbit space $\mathcal{F}/\mathsf{G}$ *i.e.* physical field space is the following

$$\gamma^\perp_{ij} = \gamma_{ij} - \gamma_{ik}K^k_\alpha N^{\alpha\beta}K^l_\beta\gamma_{lj}\,.$$

Vilkoviski Configuration Field Space Connection

The connection that is compatible with the configuration space metric i.e. which satisfies the condition

$$\nabla_i \gamma^{\perp}_{jk} = 0$$

takes the form

$$\bar{\Gamma}^{i}_{kl} = \Gamma_{kl}{}^{i} + T^{i}_{kl},$$

where $\Gamma_{kl}{}^{i}$ is the connection built with the full metric γ whereas the last term is defined as

$$T^{i}_{kl} = -2\gamma_{(k|r} K^{r}_{\alpha} N^{\alpha\beta} D_{l)} K^{s}_{\beta} + \gamma_{(k|r} K^{r}_{\alpha} N^{\alpha\beta} K^{p}_{\alpha} D_p K^{s}_{\mu} N^{\mu\nu} K^{p}_{\nu)}$$

The important property results

$$\gamma^{\perp}_{ij} K^{j}_{\alpha} = 0 = \gamma^{\perp}_{ij} \nabla_k K^{j}_{\alpha}, \Rightarrow \nabla_k K^{j}_{\alpha} \sim K^{j}_{\mu}$$

The Vilkoviski-DeWitt Effective Action reads

$$e^{-\Gamma_{\mathrm{VD}}[\bar{\phi}]} = \int \mathcal{D}\mu[\bar{\phi};\phi]\delta[\chi]\exp\left\{-S[\phi] - \sigma^i[\bar{\phi},\phi]C^{-1}{}_i^{\ j}[\bar{\phi}]\frac{\delta\Gamma_{\mathrm{VD}}[\bar{\phi}]}{\delta\bar{\phi}^j}\right\}$$

where $\sigma[\varphi,\phi] = 1/2(\text{geodesic connecting } \varphi \text{ and } \phi)^2$ and

$$\mathcal{D}\mu[\bar{\phi};\phi] = \mathcal{D}\phi\sqrt{g[\phi]}\det Q[\bar{\phi},\phi],\ C^{-1}{}^i_{\ j}[\bar{\phi}] \approx \delta^i_j + \mathcal{R}^i{}_{kjl}[\varphi]\langle\sigma^k[\bar{\phi},\phi]\sigma^l[\bar{\phi},\phi]\rangle + \dots$$

Quantity $\sigma^i[\bar{\phi},\phi] \equiv g^{ij}[\bar{\phi}]\delta\sigma[\bar{\phi},\phi]/\delta\bar{\phi}^j$ at $\bar{\phi}^i$ transforms as a vector whereas at ϕ^i as a scalar. Hence quantum gauge transformations are not affected by the presence of classical currents.

Due to the properties

$$K^k{}_\alpha[\bar{\phi}]\nabla_k\sigma^i[\bar{\phi},\phi] \sim K^k{}_\beta[\bar{\phi}], \quad \text{and} \quad K^k{}_\alpha[\phi]\frac{\delta}{\delta\phi^k}\sigma^i[\bar{\phi},\phi] \sim K^k{}_\beta[\bar{\phi}]$$

the VDEA is background gauge independet, *i. e.*

$$\Gamma_{\mathrm{VD},i}[\bar{\phi}]K^i{}_\alpha[\bar{\phi}] = 0.$$

One loop approximation to VDEA

The covariant expansion of the classical action reads

$$S[\phi] = \sum_{n\geq 0} \frac{(-1)^n}{n!} (\nabla_{i_1} \dots \nabla_{i_n} S[\bar{\phi}]) \sigma^{i_1}[\bar{\phi}, \phi] \dots \sigma^{i_n}[\bar{\phi}, \phi].$$

In the one loop approximation $C^{-1}{}^{i}_{j}[\bar{\phi}] \approx \delta^i_j$. Hence,

the one loop VDEA

$$\Gamma^{(1)}_{VD}[\bar{\phi}] = \frac{1}{2} \log\det\left(\nabla_i \nabla_j S[\bar{\phi}] + \frac{1}{\alpha} \chi^{\mu}_{,i}[\bar{\phi}] c_{\mu\nu} \chi^{\nu}_{,j}[\bar{\phi}]\right) - \log\det Q[\bar{\phi}] - \frac{1}{2} \log\det \gamma(\bar{\phi}),$$

Solution to the gauge-fixing dependence: Taking another gauge-fixing term $\chi'^{\alpha} = \chi^{\alpha} + \Delta\chi^{\alpha}$ we get

$$\delta_{\chi} \Gamma_{VD}[\bar{\phi}] = -G^{ij} S_{,k} \nabla_i K^k_{\alpha} Q^{-1\alpha}_{\beta} \Delta\chi^{\beta}_{,j} = 0\,,$$

that is the effective action is independent of the gauge fixing term.
In "the orthogonal gauge"

$$K^i_{\alpha}[\bar{\phi}] g_{ij}[\bar{\phi}] \sigma^j[\bar{\phi}, \phi] = 0$$

connection simplifies to the Christoffel one *i.e.* $\bar{\Gamma}^m_{ij} = \Gamma^m_{ij}$ and $T^i_{kl} = 0$;

The gravitational configuration field space $\mathcal{F}$ is endowed with one-parameter family of ultralocal metrics ($\varphi^i \to g_{\{x,\mu\nu\}} = g_{\mu\nu}(x)$)

$$\gamma_{ij}(\varphi;a) \to \sqrt{g(x)}\,\mathcal{G}^{\mu\nu,\alpha\beta}(x;a)\bar{\delta}(x,x') = \sqrt{g(x)}\,\frac{1}{4}\left(2g^{\mu(\alpha}g^{\beta)\nu} - a\,g^{\mu\nu}g^{\alpha\beta}\right)(x)\bar{\delta}(x,x'),$$

where $a \neq \frac{1}{2}, 2$. In the case of Einstein gravity metric can be chosen from the highest derivative term in the second order expansion of the action about a background configuration

$$S_{,ij} = \gamma_{ik}\mathcal{A}^k_j + \mathcal{C}_{ij}\nabla\nabla + \mathcal{B}_{ij} = P^k_i\gamma_{kl}P^l_j + \mathcal{B}_{ij},$$

where

$$\gamma_{ij} = \gamma_{ij}(\varphi;1), \quad \mathcal{A}^k_j \to \delta^{\alpha\beta}_{\mu'\nu'}\Box\bar{\delta}(x,x'), \quad \mathcal{B}_{ij} \sim R_{..}\text{(curvatures in background fields)}.$$

Is it possible to change parametrization s.t.

$$\gamma_{ij}(\varphi;a) = \frac{\partial\varphi'^k}{\partial\varphi^i}\frac{\partial\varphi'^l}{\partial\varphi^j}\gamma_{kl}(\varphi';1)\,?$$

ANSWER: NO.

In principle one can choose any metric on $\mathcal{F}$. However, this leads to the one-parameter dependent results when VDW effective action is used. Indeed,

$$\frac{1}{2}\log\det\Big(\nabla_i(a)\nabla_j(a)S[\bar{\phi}] + \tfrac{1}{\alpha}\chi^{\mu}_{,i}[\bar{\phi};a]\,c_{\mu\nu}\,\chi^{\nu}_{,j}[\bar{\phi};a]\Big)$$
$$= \frac{1}{2}\log\det\Big(\nabla_i\nabla_j S[\bar{\phi}] + \tfrac{1}{\alpha}\chi^{\mu}_{,i}[\bar{\phi}]\,c_{\mu\nu}\,\chi^{\nu}_{,j}[\bar{\phi}]\Big) + (a-1)\,G^{ij}P^k_i H_{kl}P^l_j + \mathcal{O}((a-1)^2)\,,$$

where

$$H_{kl} \equiv 2P^m_k\gamma_{mr}Q^r_s K^s_\alpha N^{\alpha\beta}K^p_\beta D_p D_n S P^n_j\,,$$

and

$$\gamma_{ij} = \gamma_{ik}\Pi^k_j + \gamma_{ik}Q^k_j, \quad \Pi^k_j \equiv \delta^k_j - Q^k_j, \quad Q^k_j \to \tfrac{1}{4}g_{\mu\nu}g^{\alpha\beta}.$$

Hence, H_{ij} does not vanish due to the presence of projector of symmetric tensor field on its trace Q^i_j.

The Scalar field interacting with Gravity

The action

$$S = -\frac{1}{\kappa^2}\int \mathrm{d}^n x\sqrt{\bar{g}}(\bar{R}-2\Lambda) + \int \mathrm{d}^n x\sqrt{\bar{g}}\left(\tfrac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\bar{\varphi}\partial_\nu\bar{\varphi} + V(\bar{\varphi})\right),$$

where

$$V(\bar{\varphi}) = \tfrac{1}{2}m^2\bar{\varphi}^2 + \tfrac{\lambda}{4!}\bar{\varphi}^4$$

Expansion of $S[g,\varphi]$ about the background field configuration yields:

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \kappa h_{\mu\nu} \quad \text{i} \quad \bar{\varphi} = \varphi/\kappa + \phi.$$

Hence

$$\begin{aligned} S_{,ij} \quad \to \quad & \tfrac{1}{2}h_{\mu\nu}\left(-\mathcal{G}^{\mu\nu,\alpha\beta}\Box - 2\mathcal{G}^{\mu\nu,\alpha\beta}\Lambda + X_\varphi^{\mu\nu,\alpha\beta} + X_g^{\mu\nu,\alpha\beta}\right)h_{\alpha\beta} - \tfrac{1}{2}C_\mu^2 \\ & + \tfrac{1}{2}\phi\left(-\Box + m^2 + \omega\varphi^2\right)\phi \quad \text{where} \quad \omega \equiv \lambda/2\kappa^2 \\ & - h_{\mu\nu}Q^{\mu\nu,\alpha}\nabla_\alpha\phi + h_{\mu\nu}\left(\tfrac{1}{2}V'(\varphi)g^{\mu\nu}\right)\phi. \end{aligned}$$

Metric and Connection

The metric on $\mathcal{F}$

$$\mathrm{d}s^2 = \frac{1}{\kappa^2}\int \mathrm{d}^n x\sqrt{g}\,\mathcal{G}^{\mu\nu,\rho\sigma}(a)\mathrm{d}g_{\mu\nu}(x)\mathrm{d}g_{\rho\sigma}(x) + \int \mathrm{d}^n x\sqrt{g}\,\mathrm{d}\varphi(x)\mathrm{d}\varphi(x)$$

The Christoffel connection on $\mathcal{F}$

$$\begin{aligned}
\Gamma^{\mu\nu,\rho\sigma}{}_{\alpha\beta} &= -\delta^{\mu\nu,\rho\sigma}_{\alpha\beta} + \tfrac{1}{4}(g^{\rho\sigma}\delta^{\mu\nu}_{\alpha\beta} + g^{\mu\nu}\delta^{\rho\sigma}_{\alpha\beta}) + \tfrac{1}{2(2a-1)}g_{\alpha\beta}\mathcal{G}^{\mu\nu,\rho\sigma}(a),\\
\Gamma^{\mu\nu,11}{}_{11} &= \tfrac{1}{4}g^{\mu\nu},\\
\Gamma^{11,11}{}_{\mu\nu} &= \kappa^2\frac{1}{2(2a-1)}g_{\mu\nu}
\end{aligned}$$

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The β-function for φ^4 theory (MS)

$$\beta_\lambda(g) = \frac{1}{(4\pi)^2}\left[3\lambda^2 + 2\left(g_\Lambda - 4g_m\frac{4a-3}{2a-1}\right)g_\kappa\lambda\right]$$

Anomalous dimension for the mass operator (MS)

$$\gamma_m = \frac{1}{(4\pi)^2}\left[\left(1-\frac{2g_\Lambda}{g_m}\right)\lambda + \frac{g_\kappa}{2a-1}(8g_\Lambda - 5g_m)\right].$$

VD effective action and gravitational corrections

The action for the abelian YM theory

$$S = -\frac{\mu^{n-4}}{\kappa^2}\int d^n x\sqrt{g}(R-2\Lambda) + \frac{1}{4e^2}\mu^{n-4}\int d^n x\sqrt{g}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\bar{F}_{\mu\nu}\bar{F}_{\alpha\beta}$$

Expanding about the background field configuration $\phi^i = \varphi^i + \eta^i$, where $\varphi^i = (g_{\mu\nu}, A_\alpha)$ and $\eta^i = (\kappa h_{\mu\nu}, a_\alpha)$ and taking one-parameter dependent metric on the full field space

$$ds^2 = \frac{1}{\kappa^2}\int d^n x\sqrt{g}\,\mathcal{G}^{\mu\nu,\rho\sigma}(a)dg_{\mu\nu}(x)dg_{\rho\sigma}(x) + \int d^n x\sqrt{g}\,g^{\alpha\beta}dA_\alpha(x)dA_\beta(x),$$

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$$\mathrm{d}s^2 = \frac{1}{\kappa^2}\int \mathrm{d}^n x\sqrt{g}\,\mathcal{G}^{\mu\nu,\rho\sigma}(a)\mathrm{d}g_{\mu\nu}(x)\mathrm{d}g_{\rho\sigma}(x) + \int \mathrm{d}^n x\sqrt{g}\,g^{\alpha\beta}\mathrm{d}A_\alpha(x)\mathrm{d}A_\beta(x),$$

and corresponding Christoffel connection

$$\begin{aligned}\Gamma^{\mu,\nu}{}_{\alpha\beta} &= \kappa^2\delta^{\mu\nu}_{\alpha\beta};\\ \Gamma^{\nu,\alpha\beta}{}_{\mu} &= -\frac{1}{2a-1}g_{\mu\lambda}\mathcal{G}^{\lambda\nu,\alpha\beta};\end{aligned}$$

VD EA is background field independent, therefore we take $g_{\mu\nu}\to\delta_{\mu\nu}$.
The form of the β function is

$$\beta(e) = -\frac{3}{2}\frac{a}{2a-1}\frac{\Lambda\kappa^2}{(4\pi)^2}e.$$