Superenergy tensors and their applications

José M M Senovilla

University of the Basque Country, Bilbao, Spain.

The 1st Conference of Polish Society on Relativity
Spała, 2nd July 2014

Outline

Introduction

- Equivalence principle $\implies$ geometrization of gravity $\implies$ there is no local energy-momentum tensor of the gravitational field.

Introduction

- Equivalence principle $\Longrightarrow$ geometrization of gravity $\Longrightarrow$ there is no local energy-momentum tensor of the gravitational field.
- Gravitational energy is non-localizable .

Introduction

- Equivalence principle $\Longrightarrow$ geometrization of gravity $\Longrightarrow$ there is no local energy-momentum tensor of the gravitational field.
- Gravitational energy is non-localizable .
- Nevertheless, there are local tensors describing the strength of the gravitational field.

Introduction

- Equivalence principle $\Longrightarrow$ geometrization of gravity $\Longrightarrow$ there is no local energy-momentum tensor of the gravitational field.
- Gravitational energy is non-localizable .
- Nevertheless, there are local tensors describing the strength of the gravitational field.
- the paradigmatic such tensor is the **Bel-Robinson tensor** given in 4 dimensions by

$$\mathcal{T}_{\alpha\beta\lambda\mu} = C_{\alpha\rho\lambda\sigma}C_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + C_{\alpha\rho\mu\sigma}C_{\beta}{}^{\rho}{}_{\lambda}{}^{\sigma} - \frac{1}{8}g_{\alpha\beta}g_{\lambda\mu}C_{\rho\tau\sigma\nu}C^{\rho\tau\sigma\nu}$$

Introduction

- Equivalence principle $\Longrightarrow$ geometrization of gravity $\Longrightarrow$ there is no local energy-momentum tensor of the gravitational field.
- Gravitational energy is non-localizable .
- Nevertheless, there are local tensors describing the strength of the gravitational field.
- the paradigmatic such tensor is the **Bel-Robinson tensor** given in 4 dimensions by

$$\mathcal{T}_{\alpha\beta\lambda\mu} = C_{\alpha\rho\lambda\sigma}C_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + C_{\alpha\rho\mu\sigma}C_{\beta}{}^{\rho}{}_{\lambda}{}^{\sigma} - \frac{1}{8}g_{\alpha\beta}g_{\lambda\mu}C_{\rho\tau\sigma\nu}C^{\rho\tau\sigma\nu}$$

- Here, $C_{\alpha\rho\lambda\sigma}$ is the Weyl tensor.

Introduction

- Equivalence principle $\Longrightarrow$ geometrization of gravity $\Longrightarrow$ there is no local energy-momentum tensor of the gravitational field.
- Gravitational energy is non-localizable .
- Nevertheless, there are local tensors describing the strength of the gravitational field.
- the paradigmatic such tensor is the **Bel-Robinson tensor** given in 4 dimensions by

$$\mathcal{T}_{\alpha\beta\lambda\mu} = C_{\alpha\rho\lambda\sigma} C_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + C_{\alpha\rho\mu\sigma} C_{\beta}{}^{\rho}{}_{\lambda}{}^{\sigma} - \frac{1}{8} g_{\alpha\beta} g_{\lambda\mu} C_{\rho\tau\sigma\nu} C^{\rho\tau\sigma\nu}$$

- Here, $C_{\alpha\rho\lambda\sigma}$ is the Weyl tensor.
- This formula is valid only in 4 dimensions (for general dimension see later) and can be also written as

$$\mathcal{T}_{\alpha\beta\lambda\mu} = C_{\alpha\rho\lambda\sigma} C_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + \star C_{\alpha\rho\lambda\sigma} \star C_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma}$$

where $\star$ denotes the Hodge dual

Introduction

- One can prove that (in 4 dimensions):

Introduction

- One can prove that (in 4 dimensions):
- $\mathcal{T}_{\alpha\beta\lambda\mu} = \mathcal{T}_{(\alpha\beta\lambda\mu)}$

Introduction

- One can prove that (in 4 dimensions):
- $\mathcal{T}_{\alpha\beta\lambda\mu} = \mathcal{T}_{(\alpha\beta\lambda\mu)}$
- $\mathcal{T}^{\rho}{}_{\rho\lambda\mu} = 0$

Introduction

- One can prove that (in 4 dimensions):
- $\mathcal{T}_{\alpha\beta\lambda\mu} = \mathcal{T}_{(\alpha\beta\lambda\mu)}$
- $\mathcal{T}^{\rho}{}_{\rho\lambda\mu} = 0$
- $$\mathcal{T}_{\alpha\beta\lambda\mu}\mathcal{T}_{\gamma}{}^{\beta\lambda\mu} = \frac{1}{4} g_{\alpha\gamma}\mathcal{T}_{\rho\beta\lambda\mu}\mathcal{T}^{\rho\beta\lambda\mu}$$

Introduction

- One can prove that (in 4 dimensions):
- $\mathcal{T}_{\alpha\beta\lambda\mu} = \mathcal{T}_{(\alpha\beta\lambda\mu)}$
- $\mathcal{T}^{\rho}{}_{\rho\lambda\mu} = 0$
- $$\mathcal{T}_{\alpha\beta\lambda\mu}\mathcal{T}_{\gamma}{}^{\beta\lambda\mu} = \frac{1}{4} g_{\alpha\gamma}\mathcal{T}_{\rho\beta\lambda\mu}\mathcal{T}^{\rho\beta\lambda\mu}$$
- $$\mathcal{T}_{\alpha\beta\lambda\mu}u^{\alpha}v^{\beta}w^{\lambda}z^{\mu} \geq 0$$

  for arbitrary future-pointing vectors u^{α}, v^{β}, w^{λ}, and z^{μ} (inequality is strict if all of them are timelike). This is called the Dominant property. ($\mathcal{T}_{0000} = 0 \Longrightarrow C_{\alpha\beta\lambda\mu} = 0$).

Introduction

- One can prove that (in 4 dimensions):
- $\mathcal{T}_{\alpha\beta\lambda\mu} = \mathcal{T}_{(\alpha\beta\lambda\mu)}$
- $\mathcal{T}^{\rho}{}_{\rho\lambda\mu} = 0$
- $$\mathcal{T}_{\alpha\beta\lambda\mu}\mathcal{T}_{\gamma}{}^{\beta\lambda\mu} = \frac{1}{4} g_{\alpha\gamma}\mathcal{T}_{\rho\beta\lambda\mu}\mathcal{T}^{\rho\beta\lambda\mu}$$
- $$\mathcal{T}_{\alpha\beta\lambda\mu}u^{\alpha}v^{\beta}w^{\lambda}z^{\mu} \geq 0$$
  for arbitrary future-pointing vectors u^{α}, v^{β}, w^{λ}, and z^{μ} (inequality is strict if all of them are timelike). This is called the Dominant property. ($\mathcal{T}_{0000} = 0 \Longrightarrow C_{\alpha\beta\lambda\mu} = 0$).
- $$\nabla^{\alpha}\mathcal{T}_{\alpha\beta\lambda\mu} = 0$$
  if the vacuum Einstein's field equations $R_{\beta\mu} = \Lambda g_{\beta\mu}$ hold.

Introduction

- One can prove that (in 4 dimensions):
- $\mathcal{T}_{\alpha\beta\lambda\mu} = \mathcal{T}_{(\alpha\beta\lambda\mu)}$
- $\mathcal{T}^{\rho}{}_{\rho\lambda\mu} = 0$
- $$\mathcal{T}_{\alpha\beta\lambda\mu}\mathcal{T}_{\gamma}{}^{\beta\lambda\mu} = \frac{1}{4} g_{\alpha\gamma}\mathcal{T}_{\rho\beta\lambda\mu}\mathcal{T}^{\rho\beta\lambda\mu}$$
- $$\mathcal{T}_{\alpha\beta\lambda\mu}u^{\alpha}v^{\beta}w^{\lambda}z^{\mu} \geq 0$$
  for arbitrary future-pointing vectors u^{α}, v^{β}, w^{λ}, and z^{μ} (inequality is strict if all of them are timelike). This is called the Dominant property. ($\mathcal{T}_{0000} = 0 \Longrightarrow C_{\alpha\beta\lambda\mu} = 0$).
- $$\nabla^{\alpha}\mathcal{T}_{\alpha\beta\lambda\mu} = 0$$
  if the vacuum Einstein's field equations $R_{\beta\mu} = \Lambda g_{\beta\mu}$ hold.
- This provides conserved quantities if there are (conformal) Killing vector fields.

Introduction: compare the electromagnetic field

- $T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = \frac{1}{2}\left(F_{\mu\rho}F_{\nu}{}^{\rho} + \star F_{\mu\rho}\star F_{\nu}{}^{\rho}\right)$

Introduction: compare the electromagnetic field

- $T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = \frac{1}{2}\left(F_{\mu\rho}F_{\nu}{}^{\rho} + \star F_{\mu\rho}\star F_{\nu}{}^{\rho}\right)$
- $T_{\mu\nu} = T_{\nu\mu}$

Introduction: compare the electromagnetic field

- $T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = \frac{1}{2}\left(F_{\mu\rho}F_{\nu}{}^{\rho} + \star F_{\mu\rho} \star F_{\nu}{}^{\rho}\right)$
- $T_{\mu\nu} = T_{\nu\mu}$
- $T^{\rho}{}_{\rho} = 0$

Introduction: compare the electromagnetic field

- $T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = \frac{1}{2}\left(F_{\mu\rho}F_{\nu}{}^{\rho} + \star F_{\mu\rho}\star F_{\nu}{}^{\rho}\right)$
- $T_{\mu\nu} = T_{\nu\mu}$
- $T^{\rho}{}_{\rho} = 0$
- $$T_{\mu\rho}T_{\nu}{}^{\rho} = \frac{1}{4}g_{\mu\nu}T_{\rho\sigma}T^{\rho\sigma}$$

Introduction: compare the electromagnetic field

- $T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = \frac{1}{2}\left(F_{\mu\rho}F_\nu{}^\rho + \star F_{\mu\rho}\star F_\nu{}^\rho\right)$
- $T_{\mu\nu} = T_{\nu\mu}$
- $T^\rho{}_\rho = 0$
- $$T_{\mu\rho}T_\nu{}^\rho = \frac{1}{4}g_{\mu\nu}T_{\rho\sigma}T^{\rho\sigma}$$
- $$T_{\mu\nu}u^\mu v^\nu \geq 0$$

  for arbitrary future-pointing vectors u^μ and v^ν (inequality is strict if all of them are timelike). This is the Dominant energy condition.

Introduction: compare the electromagnetic field

- $T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = \frac{1}{2}\left(F_{\mu\rho}F_{\nu}{}^{\rho} + \star F_{\mu\rho}\star F_{\nu}{}^{\rho}\right)$
- $T_{\mu\nu} = T_{\nu\mu}$
- $T^{\rho}{}_{\rho} = 0$
- $$T_{\mu\rho}T_{\nu}{}^{\rho} = \frac{1}{4}g_{\mu\nu}T_{\rho\sigma}T^{\rho\sigma}$$
- $$T_{\mu\nu}u^{\mu}v^{\nu} \geq 0$$

  for arbitrary future-pointing vectors u^{μ} and v^{ν} (inequality is strict if all of them are timelike). This is the Dominant energy condition.
- $\nabla^{\mu}T_{\mu\nu} = F_{\nu\rho}j^{\rho}$ and therefore $\nabla^{\mu}T_{\mu\nu} = 0$ if there are no charge nor currents ($j^{\mu} = 0$).

Introduction: compare the electromagnetic field

- $T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = \frac{1}{2}\left(F_{\mu\rho}F_{\nu}{}^{\rho} + \star F_{\mu\rho} \star F_{\nu}{}^{\rho}\right)$
- $T_{\mu\nu} = T_{\nu\mu}$
- $T^{\rho}{}_{\rho} = 0$
- $$T_{\mu\rho}T_{\nu}{}^{\rho} = \frac{1}{4}g_{\mu\nu}T_{\rho\sigma}T^{\rho\sigma}$$
- $$T_{\mu\nu}u^{\mu}v^{\nu} \geq 0$$

  for arbitrary future-pointing vectors u^{μ} and v^{ν} (inequality is strict if all of them are timelike). This is the Dominant energy condition.
- $\nabla^{\mu}T_{\mu\nu} = F_{\nu\rho}j^{\rho}$ and therefore $\nabla^{\mu}T_{\mu\nu} = 0$ if there are no charge nor currents ($j^{\mu} = 0$).
- This provides conserved quantities if there are (conformal) Killing vector fields.

Introduction: why Bel-Robinson?

- The existence of the Bel-Robinson tensor has been a kind of a mystery over the years

Introduction: why Bel-Robinson?

- The existence of the Bel-Robinson tensor has been a kind of a mystery over the years
- It is reminiscent of energy-momentum tensors, yet it is not such a thing –it cannot be!

Introduction: why Bel-Robinson?

- The existence of the Bel-Robinson tensor has been a kind of a mystery over the years
- It is reminiscent of energy-momentum tensors, yet it is not such a thing –it cannot be!
- It has four indices, instead of the usual pair.

Introduction: why Bel-Robinson?

- The existence of the Bel-Robinson tensor has been a kind of a mystery over the years
- It is reminiscent of energy-momentum tensors, yet it is not such a thing –it cannot be!
- It has four indices, instead of the usual pair.
- It looks related somehow to the energy-momentum properties of the the gravitational field, but its physical dimensions (L^{-4}) are wrong

Introduction: why Bel-Robinson?

- The existence of the Bel-Robinson tensor has been a kind of a mystery over the years
- It is reminiscent of energy-momentum tensors, yet it is not such a thing –it cannot be!
- It has four indices, instead of the usual pair.
- It looks related somehow to the energy-momentum properties of the the gravitational field, but its physical dimensions (L^{-4}) are wrong
- Thus, the name "super-energy" was coined by Bel.

Introduction: why Bel-Robinson?

- The existence of the Bel-Robinson tensor has been a kind of a mystery over the years
- It is reminiscent of energy-momentum tensors, yet it is not such a thing –it cannot be!
- It has four indices, instead of the usual pair.
- It looks related somehow to the energy-momentum properties of the the gravitational field, but its physical dimensions (L^{-4}) are wrong
- Thus, the name "super-energy" was coined by Bel.
- The following scheme led to a series of interesting developments

	$T_{\mu\nu}$	"superenergy"
Gravity	NO	YES
Physical fields	YES	??

Classical developments: Garecki's canonical supernergy tensors

- A unified treatment, valid for the gravitational as well as other physical fields has been elaborated and studied by Garecki.

Classical developments: Garecki's canonical supernergy tensors

- A unified treatment, valid for the gravitational as well as other physical fields has been elaborated and studied by Garecki.
- The basic idea is to consider an average, over small regions, of the relative differences between the energy-momentum pseudo-tensor values (in normal coordinates)

Classical developments: Garecki's canonical supernergy tensors

- A unified treatment, valid for the gravitational as well as other physical fields has been elaborated and studied by Garecki.
- The basic idea is to consider an average, over small regions, of the relative differences between the energy-momentum pseudo-tensor values (in normal coordinates)
- The same can be done for the relative differences of the energy-momentum tensors of non-gravitational fields.

Classical developments: Garecki's canonical supernergy tensors

- A unified treatment, valid for the gravitational as well as other physical fields has been elaborated and studied by Garecki.
- The basic idea is to consider an average, over small regions, of the relative differences between the energy-momentum pseudo-tensor values (in normal coordinates)
- The same can be done for the relative differences of the energy-momentum tensors of non-gravitational fields.
- There is a relation between this definition in the gravitational case and the Bel-Robinson tensor.

Classical developments: Bel tensor

- A first step was immediately taken by Bel himself in 1958. The Bel tensor, *including matter*:

$$
\begin{aligned}
B_{\alpha\beta\lambda\mu} \equiv\; & R_{\alpha\rho\lambda\sigma}R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + R_{\alpha\rho\mu\sigma}R_{\beta}{}^{\rho}{}_{\lambda}{}^{\sigma} - \frac{1}{2}g_{\alpha\beta}R_{\rho\tau\lambda\sigma}R^{\rho\tau}{}_{\mu}{}^{\sigma} \\
& -\frac{1}{2}g_{\lambda\mu}R_{\alpha\rho\sigma\tau}R_{\beta}{}^{\rho\sigma\tau} + \frac{1}{8}g_{\alpha\beta}g_{\lambda\mu}R_{\rho\tau\sigma\nu}R^{\rho\tau\sigma\nu}
\end{aligned}
$$

(This is valid in general dimension n. Replacing R by C one gets the Bel-Robinson in general n).

Classical developments: Bel tensor

- A first step was immediately taken by Bel himself in 1958. The Bel tensor, *including matter*:

$$\begin{aligned} B_{\alpha\beta\lambda\mu} \equiv\ & R_{\alpha\rho\lambda\sigma}R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + R_{\alpha\rho\mu\sigma}R_{\beta}{}^{\rho}{}_{\lambda}{}^{\sigma} - \frac{1}{2}g_{\alpha\beta}R_{\rho\tau\lambda\sigma}R^{\rho\tau}{}_{\mu}{}^{\sigma} \\ & -\frac{1}{2}g_{\lambda\mu}R_{\alpha\rho\sigma\tau}R_{\beta}{}^{\rho\sigma\tau} + \frac{1}{8}g_{\alpha\beta}g_{\lambda\mu}R_{\rho\tau\sigma\nu}R^{\rho\tau\sigma\nu} \end{aligned}$$

  (This is valid in general dimension n. Replacing R by C one gets the Bel-Robinson in general n).

- $B_{\alpha\beta\lambda\mu} = B_{(\alpha\beta)(\lambda\mu)} = B_{\lambda\mu\alpha\beta}$

Classical developments: Bel tensor

- A first step was immediately taken by Bel himself in 1958. The Bel tensor, *including matter*:

$$B_{\alpha\beta\lambda\mu} \equiv R_{\alpha\rho\lambda\sigma}R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + R_{\alpha\rho\mu\sigma}R_{\beta}{}^{\rho}{}_{\lambda}{}^{\sigma} - \frac{1}{2}g_{\alpha\beta}R_{\rho\tau\lambda\sigma}R^{\rho\tau}{}_{\mu}{}^{\sigma} - \frac{1}{2}g_{\lambda\mu}R_{\alpha\rho\sigma\tau}R_{\beta}{}^{\rho\sigma\tau} + \frac{1}{8}g_{\alpha\beta}g_{\lambda\mu}R_{\rho\tau\sigma\nu}R^{\rho\tau\sigma\nu}$$

  (This is valid in general dimension n. Replacing R by C one gets the Bel-Robinson in general n).
- $B_{\alpha\beta\lambda\mu} = B_{(\alpha\beta)(\lambda\mu)} = B_{\lambda\mu\alpha\beta}$
- $B^{\rho}{}_{\rho\lambda\mu} = 0$ in $n = 4$.

Classical developments: Bel tensor

- A first step was immediately taken by Bel himself in 1958. The Bel tensor, *including matter*:

$$B_{\alpha\beta\lambda\mu} \equiv R_{\alpha\rho\lambda\sigma}R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + R_{\alpha\rho\mu\sigma}R_{\beta}{}^{\rho}{}_{\lambda}{}^{\sigma} - \frac{1}{2}g_{\alpha\beta}R_{\rho\tau\lambda\sigma}R^{\rho\tau}{}_{\mu}{}^{\sigma} - \frac{1}{2}g_{\lambda\mu}R_{\alpha\rho\sigma\tau}R_{\beta}{}^{\rho\sigma\tau} + \frac{1}{8}g_{\alpha\beta}g_{\lambda\mu}R_{\rho\tau\sigma\nu}R^{\rho\tau\sigma\nu}$$

  (This is valid in general dimension n. Replacing R by C one gets the Bel-Robinson in general n).

- $B_{\alpha\beta\lambda\mu} = B_{(\alpha\beta)(\lambda\mu)} = B_{\lambda\mu\alpha\beta}$
- $B^{\rho}{}_{\rho\lambda\mu} = 0$ in $n = 4$.
- $B_{\alpha\beta\lambda\mu}u^{\alpha}v^{\beta}w^{\lambda}z^{\mu} \geq 0$ for arbitrary future-pointing vectors u^{α}, v^{β}, w^{λ}, and z^{μ} (inequality is strict if all of them are timelike).

Classical developments: Bel tensor

- A first step was immediately taken by Bel himself in 1958. The Bel tensor, *including matter*:

$$B_{\alpha\beta\lambda\mu} \equiv R_{\alpha\rho\lambda\sigma}R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + R_{\alpha\rho\mu\sigma}R_{\beta}{}^{\rho}{}_{\lambda}{}^{\sigma} - \frac{1}{2}g_{\alpha\beta}R_{\rho\tau\lambda\sigma}R^{\rho\tau}{}_{\mu}{}^{\sigma}$$
$$-\frac{1}{2}g_{\lambda\mu}R_{\alpha\rho\sigma\tau}R_{\beta}{}^{\rho\sigma\tau} + \frac{1}{8}g_{\alpha\beta}g_{\lambda\mu}R_{\rho\tau\sigma\nu}R^{\rho\tau\sigma\nu}$$

  (This is valid in general dimension n. Replacing R by C one gets the Bel-Robinson in general n).

- $B_{\alpha\beta\lambda\mu} = B_{(\alpha\beta)(\lambda\mu)} = B_{\lambda\mu\alpha\beta}$
- $B^{\rho}{}_{\rho\lambda\mu} = 0$ in $n = 4$.
- $B_{\alpha\beta\lambda\mu}u^{\alpha}v^{\beta}w^{\lambda}z^{\mu} \geq 0$ for arbitrary future-pointing vectors u^{α}, v^{β}, w^{λ}, and z^{μ} (inequality is strict if all of them are timelike).
- $\nabla_{\alpha}B^{\alpha\beta\lambda\mu} = R^{\beta}_{\rho}{}^{\lambda}{}_{\sigma}J^{\mu\sigma\rho} + R^{\beta}_{\rho}{}^{\mu}{}_{\sigma}J^{\lambda\sigma\rho} - \frac{1}{2}g^{\lambda\mu}R^{\beta}_{\rho\sigma\gamma}J^{\sigma\gamma\rho}$ where $J_{\lambda\mu\beta} \equiv \nabla_{\lambda}R_{\mu\beta} - \nabla_{\mu}R_{\lambda\beta}$ (Compare with $\nabla^{\mu}T_{\mu\nu} = F_{\nu\rho}j^{\rho}$).

Next step: Chevreton tensor

The next step was taken by Chevreton in 1964, who tried to define the super-energy tensor of the electromagnetic field.

Next step: Chevreton tensor

The next step was taken by Chevreton in 1964, who tried to define the super-energy tensor of the electromagnetic field.

$$
\begin{aligned}
H_{\alpha\beta\lambda\mu} = \frac{1}{2}\Big[& \nabla_\alpha F_{\lambda\rho} \nabla_\beta F_\mu{}^\rho + \nabla_\lambda F_{\alpha\rho} \nabla_\mu F_\beta{}^\rho \\
& + \nabla_\lambda F_{\beta\rho} \nabla_\mu F_\alpha{}^\rho + \nabla_\alpha F_{\mu\rho} \nabla_\beta F_\lambda{}^\rho \\
& - g_{\alpha\beta}\left(\nabla_\sigma F_{\lambda\rho} \nabla^\sigma F_\mu{}^\rho + \frac{1}{2} \nabla_\lambda F_{\sigma\rho} \nabla_\mu F^{\sigma\rho} \right) \\
& - g_{\lambda\mu}\left(\nabla_\sigma F_{\alpha\rho} \nabla^\sigma F_\beta{}^\rho + \frac{1}{2} \nabla_\alpha F_{\sigma\rho} \nabla_\beta F^{\sigma\rho} \right) \\
& + \frac{1}{2} g_{\alpha\beta} g_{\lambda\mu} \nabla_\tau F_{\sigma\rho} \nabla^\tau F^{\sigma\rho} \Big]
\end{aligned}
$$

Next step: Chevreton tensor

The next step was taken by Chevreton in 1964, who tried to define the super-energy tensor of the electromagnetic field.

$$
\begin{aligned}
H_{\alpha\beta\lambda\mu} = \frac{1}{2}\Big[& \nabla_\alpha F_{\lambda\rho}\nabla_\beta F_\mu{}^\rho + \nabla_\lambda F_{\alpha\rho}\nabla_\mu F_\beta{}^\rho \\
& + \nabla_\lambda F_{\beta\rho}\nabla_\mu F_\alpha{}^\rho + \nabla_\alpha F_{\mu\rho}\nabla_\beta F_\lambda{}^\rho \\
& - g_{\alpha\beta}\left(\nabla_\sigma F_{\lambda\rho}\nabla^\sigma F_\mu{}^\rho + \frac{1}{2}\nabla_\lambda F_{\sigma\rho}\nabla_\mu F^{\sigma\rho}\right) \\
& - g_{\lambda\mu}\left(\nabla_\sigma F_{\alpha\rho}\nabla^\sigma F_\beta{}^\rho + \frac{1}{2}\nabla_\alpha F_{\sigma\rho}\nabla_\beta F^{\sigma\rho}\right) \\
& + \frac{1}{2} g_{\alpha\beta} g_{\lambda\mu}\nabla_\tau F_{\sigma\rho}\nabla^\tau F^{\sigma\rho}\Big]
\end{aligned}
$$

(This is valid in general dimension n).

Properties of Chevreton tensor

- $H_{\alpha\beta\lambda\mu} = H_{(\alpha\beta)(\lambda\mu)} = H_{\lambda\mu\alpha\beta}$. Actually, $H_{\alpha\beta\lambda\mu} = H_{(\alpha\beta\lambda\mu)}$ in $n = 4$.

Properties of Chevreton tensor

- $H_{\alpha\beta\lambda\mu} = H_{(\alpha\beta)(\lambda\mu)} = H_{\lambda\mu\alpha\beta}$. Actually, $H_{\alpha\beta\lambda\mu} = H_{(\alpha\beta\lambda\mu)}$ in $n = 4$.
- $H^{\rho\sigma}{}_{\rho\sigma} = 0$ in $n = 4$.

Properties of Chevreton tensor

- $H_{\alpha\beta\lambda\mu} = H_{(\alpha\beta)(\lambda\mu)} = H_{\lambda\mu\alpha\beta}$. Actually, $H_{\alpha\beta\lambda\mu} = H_{(\alpha\beta\lambda\mu)}$ in $n = 4$.
- $H^{\rho\sigma}{}_{\rho\sigma} = 0$ in $n = 4$.
- $H_{\alpha\beta\lambda\mu}u^{\alpha}v^{\beta}w^{\lambda}z^{\mu} \geq 0$ for arbitrary future-pointing vectors u^{α}, v^{β}, w^{λ}, and z^{μ} (inequality is strict if all of them are timelike).

Properties of Chevreton tensor

- $H_{\alpha\beta\lambda\mu} = H_{(\alpha\beta)(\lambda\mu)} = H_{\lambda\mu\alpha\beta}$. Actually, $H_{\alpha\beta\lambda\mu} = H_{(\alpha\beta\lambda\mu)}$ in $n = 4$.
- $H^{\rho\sigma}{}_{\rho\sigma} = 0$ in $n = 4$.
- $H_{\alpha\beta\lambda\mu} u^\alpha v^\beta w^\lambda z^\mu \geq 0$ for arbitrary future-pointing vectors u^α, v^β, w^λ, and z^μ (inequality is strict if all of them are timelike).
- $\nabla_\alpha H^{\alpha\beta\lambda\mu} \neq 0$. (Long expression)

Properties of Chevreton tensor

- $H_{\alpha\beta\lambda\mu} = H_{(\alpha\beta)(\lambda\mu)} = H_{\lambda\mu\alpha\beta}$. Actually, $H_{\alpha\beta\lambda\mu} = H_{(\alpha\beta\lambda\mu)}$ in $n = 4$.
- $H^{\rho\sigma}{}_{\rho\sigma} = 0$ in $n = 4$.
- $H_{\alpha\beta\lambda\mu} u^\alpha v^\beta w^\lambda z^\mu \geq 0$ for arbitrary future-pointing vectors u^α, v^β, w^λ, and z^μ (inequality is strict if all of them are timelike).
- $\nabla_\alpha H^{\alpha\beta\lambda\mu} \neq 0$. (Long expression)
- However, $\nabla_\alpha H^{\alpha\beta\lambda\mu} = 0$ in flat spacetime!

Properties of Chevreton tensor

- $H_{\alpha\beta\lambda\mu} = H_{(\alpha\beta)(\lambda\mu)} = H_{\lambda\mu\alpha\beta}$. Actually, $H_{\alpha\beta\lambda\mu} = H_{(\alpha\beta\lambda\mu)}$ in $n = 4$.
- $H^{\rho\sigma}{}_{\rho\sigma} = 0$ in $n = 4$.
- $H_{\alpha\beta\lambda\mu} u^\alpha v^\beta w^\lambda z^\mu \geq 0$ for arbitrary future-pointing vectors u^α, v^β, w^λ, and z^μ (inequality is strict if all of them are timelike).
- $\nabla_\alpha H^{\alpha\beta\lambda\mu} \neq 0$. (Long expression)
- However, $\nabla_\alpha H^{\alpha\beta\lambda\mu} = 0$ in flat spacetime!
- In other words: $H_{\alpha\beta\lambda\mu}$ leads to conserved quantities in the absence of gravitation. Recall that $B_{\alpha\beta\lambda\mu}$ led to conservation currents in the absence of fields...

A general superenergy construction

- In 1999 I introduced a general definition of "super energy tensor" (see CQG **17** (2000) 2799-2842)

A general superenergy construction

- In 1999 I introduced a general definition of "super energy tensor" (see CQG **17** (2000) 2799-2842)
- Given **any** tensor field A, this provides the (essentially unique) tensor $T\{A\}$ quadratic in A with the dominant property.

A general superenergy construction

- In 1999 I introduced a general definition of "super energy tensor" (see CQG **17** (2000) 2799-2842)
- Given **any** tensor field A, this provides the (essentially unique) tensor $T\{A\}$ quadratic in A with the dominant property.
- This definition recovers the energy-momentum tensor of classical fields, as well as the Bel-Robinson, Bel, and Chevreton tensors:

A general superenergy construction

- In 1999 I introduced a general definition of "super energy tensor" (see CQG **17** (2000) 2799-2842)
- Given **any** tensor field A, this provides the (essentially unique) tensor $T\{A\}$ quadratic in A with the dominant property.
- This definition recovers the energy-momentum tensor of classical fields, as well as the Bel-Robinson, Bel, and Chevreton tensors:
  1. electromagnetic field: the tensor A is $F_{\mu\nu}$

A general superenergy construction

- In 1999 I introduced a general definition of "super energy tensor" (see CQG **17** (2000) 2799-2842)
- Given **any** tensor field A, this provides the (essentially unique) tensor $T\{A\}$ quadratic in A with the dominant property.
- This definition recovers the energy-momentum tensor of classical fields, as well as the Bel-Robinson, Bel, and Chevreton tensors:
  1. electromagnetic field: the tensor A is $F_{\mu\nu}$
  2. scalar field: the tensor A is $\nabla_\mu \phi$,

A general superenergy construction

- In 1999 I introduced a general definition of "super energy tensor" (see CQG **17** (2000) 2799-2842)
- Given **any** tensor field A, this provides the (essentially unique) tensor $T\{A\}$ quadratic in A with the dominant property.
- This definition recovers the energy-momentum tensor of classical fields, as well as the Bel-Robinson, Bel, and Chevreton tensors:
  1. electromagnetic field: the tensor A is $F_{\mu\nu}$
  2. scalar field: the tensor A is $\nabla_\mu \phi$,
  3. for Bel-Robinson: the tensor A is $C_{\alpha\beta\mu\nu}$

A general superenergy construction

- In 1999 I introduced a general definition of "super energy tensor" (see CQG **17** (2000) 2799-2842)
- Given **any** tensor field A, this provides the (essentially unique) tensor $T\{A\}$ quadratic in A with the dominant property.
- This definition recovers the energy-momentum tensor of classical fields, as well as the Bel-Robinson, Bel, and Chevreton tensors:
  1. electromagnetic field: the tensor A is $F_{\mu\nu}$
  2. scalar field: the tensor A is $\nabla_\mu \phi$,
  3. for Bel-Robinson: the tensor A is $C_{\alpha\beta\mu\nu}$
  4. for Bel: the tensor A is $R_{\alpha\beta\mu\nu}$

A general superenergy construction

- In 1999 I introduced a general definition of "super energy tensor" (see CQG **17** (2000) 2799-2842)
- Given **any** tensor field A, this provides the (essentially unique) tensor $T\{A\}$ quadratic in A with the dominant property.
- This definition recovers the energy-momentum tensor of classical fields, as well as the Bel-Robinson, Bel, and Chevreton tensors:
  1. electromagnetic field: the tensor A is $F_{\mu\nu}$
  2. scalar field: the tensor A is $\nabla_\mu \phi$,
  3. for Bel-Robinson: the tensor A is $C_{\alpha\beta\mu\nu}$
  4. for Bel: the tensor A is $R_{\alpha\beta\mu\nu}$
  5. for Chevreton the tensor A is $\nabla_\lambda F_{\mu\nu}$

An example: massless scalar field

- Let ϕ be a massless scalar field.

An example: massless scalar field

- Let ϕ be a massless scalar field.
- using $\nabla_\mu \phi$ as seed tensor A the super-energy construction provides the standard energy-momentum tensor

$$T_{\lambda\mu}\{\nabla\phi\} = \nabla_\lambda\phi\nabla_\mu\phi - \frac{1}{2}g_{\lambda\mu}\nabla_\rho\phi\nabla^\rho\phi$$

An example: massless scalar field

- Let ϕ be a massless scalar field.
- using $\nabla_\mu \phi$ as seed tensor A the super-energy construction provides the standard energy-momentum tensor

$$T_{\lambda\mu}\{\nabla\phi\} = \nabla_\lambda \phi \nabla_\mu \phi - \frac{1}{2} g_{\lambda\mu} \nabla_\rho \phi \nabla^\rho \phi$$

- $T_{\lambda\mu}\{\nabla\phi\} = T_{\mu\lambda}\{\nabla\phi\}$

An example: massless scalar field

- Let ϕ be a massless scalar field.
- using $\nabla_\mu \phi$ as seed tensor A the super-energy construction provides the standard energy-momentum tensor

$$T_{\lambda\mu}\{\nabla\phi\} = \nabla_\lambda \phi \nabla_\mu \phi - \frac{1}{2} g_{\lambda\mu} \nabla_\rho \phi \nabla^\rho \phi$$

- $T_{\lambda\mu}\{\nabla\phi\} = T_{\mu\lambda}\{\nabla\phi\}$
- $T_{\lambda\mu}\{\nabla\phi\}$ satisfies the dominant energy condition.

An example: massless scalar field

- Let ϕ be a massless scalar field.
- using $\nabla_\mu \phi$ as seed tensor A the super-energy construction provides the standard energy-momentum tensor

$$T_{\lambda\mu}\{\nabla\phi\} = \nabla_\lambda\phi\nabla_\mu\phi - \frac{1}{2}g_{\lambda\mu}\nabla_\rho\phi\nabla^\rho\phi$$

- $T_{\lambda\mu}\{\nabla\phi\} = T_{\mu\lambda}\{\nabla\phi\}$
- $T_{\lambda\mu}\{\nabla\phi\}$ satisfies the dominant energy condition.
- $\nabla^\mu T_{\mu\nu}\{\nabla\phi\} = 0$ if the field equation $\Box\phi = 0$ holds for ϕ.

An example: superenergy tensor of scalar field

- Using instead $\nabla_\mu \nabla_\nu \phi$ as seed tensor A one gets its super-energy tensor :

$$S_{\alpha\beta\lambda\mu} = \nabla_\alpha \nabla_\lambda \phi \nabla_\mu \nabla_\beta \phi + \nabla_\alpha \nabla_\mu \phi \nabla_\lambda \nabla_\beta \phi - \\ - g_{\alpha\beta} \nabla_\lambda \nabla^\rho \phi \nabla_\mu \nabla_\rho \phi - g_{\lambda\mu} \nabla_\alpha \nabla^\rho \phi \nabla_\beta \nabla_\rho \phi \\ + \frac{1}{2} g_{\alpha\beta} g_{\lambda\mu} \nabla_\sigma \nabla_\rho \phi \nabla^\sigma \nabla^\rho \phi \,.$$

An example: superenergy tensor of scalar field

- Using instead $\nabla_\mu \nabla_\nu \phi$ as seed tensor A one gets its super-energy tensor :

$$S_{\alpha\beta\lambda\mu} = \nabla_\alpha \nabla_\lambda \phi \nabla_\mu \nabla_\beta \phi + \nabla_\alpha \nabla_\mu \phi \nabla_\lambda \nabla_\beta \phi - \\ -g_{\alpha\beta} \nabla_\lambda \nabla^\rho \phi \nabla_\mu \nabla_\rho \phi - g_{\lambda\mu} \nabla_\alpha \nabla^\rho \phi \nabla_\beta \nabla_\rho \phi \\ +\frac{1}{2} g_{\alpha\beta} g_{\lambda\mu} \nabla_\sigma \nabla_\rho \phi \nabla^\sigma \nabla^\rho \phi \,.$$

- $S_{\alpha\beta\lambda\mu} = S_{(\alpha\beta)(\lambda\mu)} = S_{\lambda\mu\alpha\beta}$

An example: superenergy tensor of scalar field

- Using instead $\nabla_\mu \nabla_\nu \phi$ as seed tensor A one gets its super-energy tensor :

$$S_{\alpha\beta\lambda\mu} = \nabla_\alpha \nabla_\lambda \phi \nabla_\mu \nabla_\beta \phi + \nabla_\alpha \nabla_\mu \phi \nabla_\lambda \nabla_\beta \phi - \\ - g_{\alpha\beta} \nabla_\lambda \nabla^\rho \phi \nabla_\mu \nabla_\rho \phi - g_{\lambda\mu} \nabla_\alpha \nabla^\rho \phi \nabla_\beta \nabla_\rho \phi \\ + \frac{1}{2} g_{\alpha\beta} g_{\lambda\mu} \nabla_\sigma \nabla_\rho \phi \nabla^\sigma \nabla^\rho \phi \, .$$

- $S_{\alpha\beta\lambda\mu} = S_{(\alpha\beta)(\lambda\mu)} = S_{\lambda\mu\alpha\beta}$
- One can actually use $S_{(\alpha\beta\lambda\mu)}$ without loss of physical generality (then, it is uniquely defined).

An example: superenergy tensor of scalar field

- Using instead $\nabla_\mu \nabla_\nu \phi$ as seed tensor A one gets its super-energy tensor :

$$S_{\alpha\beta\lambda\mu} = \nabla_\alpha \nabla_\lambda \phi \nabla_\mu \nabla_\beta \phi + \nabla_\alpha \nabla_\mu \phi \nabla_\lambda \nabla_\beta \phi - \\ -g_{\alpha\beta} \nabla_\lambda \nabla^\rho \phi \nabla_\mu \nabla_\rho \phi - g_{\lambda\mu} \nabla_\alpha \nabla^\rho \phi \nabla_\beta \nabla_\rho \phi \\ + \frac{1}{2} g_{\alpha\beta} g_{\lambda\mu} \nabla_\sigma \nabla_\rho \phi \nabla^\sigma \nabla^\rho \phi \,.$$

- $S_{\alpha\beta\lambda\mu} = S_{(\alpha\beta)(\lambda\mu)} = S_{\lambda\mu\alpha\beta}$
- One can actually use $S_{(\alpha\beta\lambda\mu)}$ without loss of physical generality (then, it is uniquely defined).
- In general

$$\nabla_\alpha S^\alpha{}_{\beta\lambda\mu} = 2\nabla_\beta \nabla_{(\lambda} \phi R_{\mu)\rho} \nabla^\rho \phi - g_{\lambda\mu} R^{\sigma\rho} \nabla_\beta \nabla_\rho \phi \nabla_\sigma \phi - \\ -\nabla_\sigma \phi \left(2\nabla^\rho \nabla_{(\lambda} \phi \, R^\sigma{}_{\mu)\rho\beta} + g_{\lambda\mu} R^\sigma{}_{\rho\beta\tau} \nabla^\rho \nabla^\tau \phi \right)$$

An example: superenergy tensor of scalar field

- Using instead $\nabla_\mu \nabla_\nu \phi$ as seed tensor A one gets its super-energy tensor :

$$S_{\alpha\beta\lambda\mu} = \nabla_\alpha \nabla_\lambda \phi \nabla_\mu \nabla_\beta \phi + \nabla_\alpha \nabla_\mu \phi \nabla_\lambda \nabla_\beta \phi - \\ - g_{\alpha\beta} \nabla_\lambda \nabla^\rho \phi \nabla_\mu \nabla_\rho \phi - g_{\lambda\mu} \nabla_\alpha \nabla^\rho \phi \nabla_\beta \nabla_\rho \phi \\ + \frac{1}{2} g_{\alpha\beta} g_{\lambda\mu} \nabla_\sigma \nabla_\rho \phi \nabla^\sigma \nabla^\rho \phi \,.$$

- $S_{\alpha\beta\lambda\mu} = S_{(\alpha\beta)(\lambda\mu)} = S_{\lambda\mu\alpha\beta}$
- One can actually use $S_{(\alpha\beta\lambda\mu)}$ without loss of physical generality (then, it is uniquely defined).
- In general

$$\nabla_\alpha S^\alpha{}_{\beta\lambda\mu} = 2\nabla_\beta \nabla_{(\lambda} \phi R_{\mu)\rho} \nabla^\rho \phi - g_{\lambda\mu} R^{\sigma\rho} \nabla_\beta \nabla_\rho \phi \nabla_\sigma \phi - \\ - \nabla_\sigma \phi \left(2\nabla^\rho \nabla_{(\lambda} \phi \, R^\sigma{}_{\mu)\rho\beta} + g_{\lambda\mu} R^\sigma{}_{\rho\beta\tau} \nabla^\rho \nabla^\tau \phi \right)$$

- Again, $S_{\alpha\beta\lambda\mu}$ is divergence-free in flat space-time, in the absence of gravitational field.

Physical considerations: units of superenergy tensors

- At the beginning, there was some confusion about the proper physical units of the Bel-Robinson tensor

Physical considerations: units of superenergy tensors

- At the beginning, there was some confusion about the proper physical units of the Bel-Robinson tensor
- Its geometrical version has units of L^{-4}, so it looks like "energy density square"

Physical considerations: units of superenergy tensors

- At the beginning, there was some confusion about the proper physical units of the Bel-Robinson tensor
- Its geometrical version has units of L^{-4}, so it looks like "energy density square"
- One could actually think that this is a "square of an energy-momentum tensor" —and this would actually explain the "positivity"—. (Observe that some terms in the Bel tensor are of type "Ricci^2" ...

Physical considerations: units of superenergy tensors

- At the beginning, there was some confusion about the proper physical units of the Bel-Robinson tensor
- Its geometrical version has units of L^{-4}, so it looks like "energy density square"
- One could actually think that this is a "square of an energy-momentum tensor" —and this would actually explain the "positivity"—. (Observe that some terms in the Bel tensor are of type "Ricci^2" ...
- Nevertheless, this is not the right answer. The correct possibility comes from splitting the L^{-4} into one energy-density and a "pure L^{-2}".

Physical considerations: units of superenergy tensors

- At the beginning, there was some confusion about the proper physical units of the Bel-Robinson tensor
- Its geometrical version has units of L^{-4}, so it looks like "energy density square"
- One could actually think that this is a "square of an energy-momentum tensor" —and this would actually explain the "positivity"—. (Observe that some terms in the Bel tensor are of type "Ricci^2" ...
- Nevertheless, this is not the right answer. The correct possibility comes from splitting the L^{-4} into one energy-density and a "pure L^{-2}".
- The justification comes from the following facts:

Units of Bel-Robinson: the small sphere limit

- A first justification comes from the afore mentioned Garecki approach.

Units of Bel-Robinson: the small sphere limit

- A first justification comes from the afore mentioned Garecki approach.
- Also: use any of the (many) definitions of quasilocal energy E and apply to a very small sphere of radius r. Then one can prove that at first non-trivial order in r one gets

$$E = \frac{4\pi}{3} r^3 T_{00} + O(r^4)$$

where T_{00} is the timelike component of the energy-momentum tensor (in a basis with $\vec{e}_0$ orthogonal to the sphere).

Units of Bel-Robinson: the small sphere limit

- A first justification comes from the afore mentioned Garecki approach.
- Also: use any of the (many) definitions of quasilocal energy E and apply to a very small sphere of radius r. Then one can prove that at first non-trivial order in r one gets

$$E = \frac{4\pi}{3} r^3 T_{00} + O(r^4)$$

where T_{00} is the timelike component of the energy-momentum tensor (in a basis with $\vec{e}_0$ orthogonal to the sphere).
- But, what happens if we are in vacuum? That is, if $T_{\mu\nu} = 0$.

Units of Bel-Robinson: the small sphere limit

- A first justification comes from the afore mentioned Garecki approach.
- Also: use any of the (many) definitions of quasilocal energy E and apply to a very small sphere of radius r. Then one can prove that at first non-trivial order in r one gets

$$E = \frac{4\pi}{3} r^3 T_{00} + O(r^4)$$

  where T_{00} is the timelike component of the energy-momentum tensor (in a basis with $\vec{e}_0$ orthogonal to the sphere).
- But, what happens if we are in vacuum? That is, if $T_{\mu\nu} = 0$.
- Then, as first proven by Horowitz and Schmidt (1982)

$$E = (const.) r^5 \mathcal{T}_{0000} + O(r^6)$$

  where $\mathcal{T}_{0000}$ is the timelike component of the Bel-Robinson tensor (the "super-energy density").

Units of superenergy

- Comparing both expressions, one derives $[\mathcal{T}] = ML^{-3}T^{-2}$

Units of superenergy

- Comparing both expressions, one derives $[\mathcal{T}] = ML^{-3}T^{-2}$
- Thus, the physical super-energy tensor should be

$$\frac{c^4}{G}\mathcal{T}_{\alpha\beta\mu\nu}$$

Units of superenergy

- Comparing both expressions, one derives $[\mathcal{T}] = ML^{-3}T^{-2}$
- Thus, the physical super-energy tensor should be

$$\frac{c^4}{G}\mathcal{T}_{\alpha\beta\mu\nu}$$

- Analogously, the gravitational energy-momentum vector of a small sphere leads to T_{0i} and, in vacuum, to $\mathcal{T}_{000i}$. The energy flux of a gravitational plane wave, for instance, travels in the direction of $\mathcal{T}_{000i}$.

Units of superenergy

- Comparing both expressions, one derives $[\mathcal{T}] = ML^{-3}T^{-2}$
- Thus, the physical super-energy tensor should be

$$\frac{c^4}{G}\mathcal{T}_{\alpha\beta\mu\nu}$$

- Analogously, the gravitational energy-momentum vector of a small sphere leads to T_{0i} and, in vacuum, to $\mathcal{T}_{000i}$. The energy flux of a gravitational plane wave, for instance, travels in the direction of $\mathcal{T}_{000i}$.
- Yet another, third, independent justification comes from the work by Teyssandier (2000), who proved that the super-energy of a quantized scalar field is interchanged in quanta of

$$\hbar\omega_k^3/c^2$$

where ω_k is the frequency of the k-mode.

Units of superenergy

- Comparing both expressions, one derives $[\mathcal{T}] = ML^{-3}T^{-2}$
- Thus, the physical super-energy tensor should be

$$\frac{c^4}{G}\mathcal{T}_{\alpha\beta\mu\nu}$$

- Analogously, the gravitational energy-momentum vector of a small sphere leads to T_{0i} and, in vacuum, to $\mathcal{T}_{000i}$. The energy flux of a gravitational plane wave, for instance, travels in the direction of $\mathcal{T}_{000i}$.
- Yet another, third, independent justification comes from the work by Teyssandier (2000), who proved that the super-energy of a quantized scalar field is interchanged in quanta of

$$\hbar\omega_k^3/c^2$$

where ω_k is the frequency of the k-mode.
- Finally, the fact that the super-energy tensor of physical fields contains two extra ∇_μ with respect to the corresponding $T_{\mu\nu}$ supports this result.

Further comments

- As we have seen, the super-energy tensors for physical fields can be put in physical correspondence with the super-energy tensor of the gravitational field.

Further comments

- As we have seen, the super-energy tensors for physical fields can be put in physical correspondence with the super-energy tensor of the gravitational field.
- They are at the same "level", carrying physical units of energy density per unit surface

Further comments

- As we have seen, the super-energy tensors for physical fields can be put in physical correspondence with the super-energy tensor of the gravitational field.
- They are at the same “level”, carrying physical units of energy density per unit surface
- The analysis of the ‘strength’ of a field at points where its energy density vanishes but such that every neighbourhood of that point contains the field requires the super-energy concept

Further comments

- As we have seen, the super-energy tensors for physical fields can be put in physical correspondence with the super-energy tensor of the gravitational field.
- They are at the same "level", carrying physical units of energy density per unit surface
- The analysis of the 'strength' of a field at points where its energy density vanishes but such that every neighbourhood of that point contains the field requires the super-energy concept
- This is why the Bel-Robinson tensor arises *naturally* in General Relativity, where the energy density of the gravitational field can be always made to vanish at **any point** by appropriate choice of the reference system — due to the equivalence principle

Further comments

- As we have seen, the super-energy tensors for physical fields can be put in physical correspondence with the super-energy tensor of the gravitational field.
- They are at the same "level", carrying physical units of energy density per unit surface
- The analysis of the 'strength' of a field at points where its energy density vanishes but such that every neighbourhood of that point contains the field requires the super-energy concept
- This is why the Bel-Robinson tensor arises *naturally* in General Relativity, where the energy density of the gravitational field can be always made to vanish at **any point** by appropriate choice of the reference system — due to the equivalence principle
- Analogously, the wave-fronts, shock waves, and similar propagating discontinuities can be properly analyzed from the super-energy viewpoint

Applications

- The super-energy tensors, and in particular the Bel-Robinson tensor, have been successfully used in many different applications, and arises as a relevant tool in many mathematical formalisms involving the gravitational field.

Applications

- The super-energy tensors, and in particular the Bel-Robinson tensor, have been successfully used in many different applications, and arises as a relevant tool in many mathematical formalisms involving the gravitational field.
- Outstanding cases are:

Applications

- The super-energy tensors, and in particular the Bel-Robinson tensor, have been successfully used in many different applications, and arises as a relevant tool in many mathematical formalisms involving the gravitational field.
- Outstanding cases are:
  1. the hyperbolic formulations of the Einstein field equations,

Applications

- The super-energy tensors, and in particular the Bel-Robinson tensor, have been successfully used in many different applications, and arises as a relevant tool in many mathematical formalisms involving the gravitational field.
- Outstanding cases are:
    1. the hyperbolic formulations of the Einstein field equations,
    2. the causal propagation of gravity and other fields,

Applications

- The super-energy tensors, and in particular the Bel-Robinson tensor, have been successfully used in many different applications, and arises as a relevant tool in many mathematical formalisms involving the gravitational field.
- Outstanding cases are:
    1. the hyperbolic formulations of the Einstein field equations,
    2. the causal propagation of gravity and other fields,
    3. the existence of global solutions of the Cauchy problem

Applications

- The super-energy tensors, and in particular the Bel-Robinson tensor, have been successfully used in many different applications, and arises as a relevant tool in many mathematical formalisms involving the gravitational field.
- Outstanding cases are:
  1. the hyperbolic formulations of the Einstein field equations,
  2. the causal propagation of gravity and other fields,
  3. the existence of global solutions of the Cauchy problem
  4. the study of the global stability of spacetimes.

Applications

- The super-energy tensors, and in particular the Bel-Robinson tensor, have been successfully used in many different applications, and arises as a relevant tool in many mathematical formalisms involving the gravitational field.
- Outstanding cases are:
  1. the hyperbolic formulations of the Einstein field equations,
  2. the causal propagation of gravity and other fields,
  3. the existence of global solutions of the Cauchy problem
  4. the study of the global stability of spacetimes.
  5. Rainich-like conditions

Applications

- The super-energy tensors, and in particular the Bel-Robinson tensor, have been successfully used in many different applications, and arises as a relevant tool in many mathematical formalisms involving the gravitational field.
- Outstanding cases are:
  1. the hyperbolic formulations of the Einstein field equations,
  2. the causal propagation of gravity and other fields,
  3. the existence of global solutions of the Cauchy problem
  4. the study of the global stability of spacetimes.
  5. Rainich-like conditions
  6. Causal (future and past) tensors

Applications

- The super-energy tensors, and in particular the Bel-Robinson tensor, have been successfully used in many different applications, and arises as a relevant tool in many mathematical formalisms involving the gravitational field.
- Outstanding cases are:
  1. the hyperbolic formulations of the Einstein field equations,
  2. the causal propagation of gravity and other fields,
  3. the existence of global solutions of the Cauchy problem
  4. the study of the global stability of spacetimes.
  5. Rainich-like conditions
  6. Causal (future and past) tensors
  7. Propagation of fields discontinuities (characteristics and bi-characteristics)

Applications

- The super-energy tensors, and in particular the Bel-Robinson tensor, have been successfully used in many different applications, and arises as a relevant tool in many mathematical formalisms involving the gravitational field.
- Outstanding cases are:
  1. the hyperbolic formulations of the Einstein field equations,
  2. the causal propagation of gravity and other fields,
  3. the existence of global solutions of the Cauchy problem
  4. the study of the global stability of spacetimes.
  5. Rainich-like conditions
  6. Causal (future and past) tensors
  7. Propagation of fields discontinuities (characteristics and bi-characteristics)
  8. Supergravity, string theory and all that...

Applications

- The super-energy tensors, and in particular the Bel-Robinson tensor, have been successfully used in many different applications, and arises as a relevant tool in many mathematical formalisms involving the gravitational field.
- Outstanding cases are:
  1. the hyperbolic formulations of the Einstein field equations,
  2. the causal propagation of gravity and other fields,
  3. the existence of global solutions of the Cauchy problem
  4. the study of the global stability of spacetimes.
  5. Rainich-like conditions
  6. Causal (future and past) tensors
  7. Propagation of fields discontinuities (characteristics and bi-characteristics)
  8. Supergravity, string theory and all that...
  9. others to be detailed presently

Application: new conservation laws for $F_{\mu\nu}$

- Consider the trace of the Chevreton tensor $H_{\mu\nu} = H^{\rho}{}_{\rho\mu\nu}$.

Application: new conservation laws for $F_{\mu\nu}$

- Consider the trace of the Chevreton tensor $H_{\mu\nu} = H^{\rho}{}_{\rho\mu\nu}$.
- We have

$$H_{\mu\nu} = \nabla_\tau F_{\mu\rho} \nabla^\tau F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} \nabla_\tau F_{\rho\sigma} \nabla^\tau F^{\rho\sigma} =$$

$$= \frac{1}{2} \left(\nabla_\tau F_{\mu\rho} \nabla^\tau F_\nu{}^\rho + \nabla_\tau \star F_{\mu\rho} \nabla^\tau \star F_\nu{}^\rho \right)$$

Application: new conservation laws for $F_{\mu\nu}$

- Consider the trace of the Chevreton tensor $H_{\mu\nu} = H^{\rho}{}_{\rho\mu\nu}$.
- We have

$$H_{\mu\nu} = \nabla_\tau F_{\mu\rho} \nabla^\tau F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} \nabla_\tau F_{\rho\sigma} \nabla^\tau F^{\rho\sigma} = \\ = \frac{1}{2} \left(\nabla_\tau F_{\mu\rho} \nabla^\tau F_\nu{}^\rho + \nabla_\tau \star F_{\mu\rho} \nabla^\tau \star F_\nu{}^\rho \right)$$

- Thus, $H_{\mu\nu} = H_{\nu\mu}$ and $H^{\rho}{}_{\rho} = 0$.

Application: new conservation laws for $F_{\mu\nu}$

- Consider the trace of the Chevreton tensor $H_{\mu\nu} = H^{\rho}{}_{\rho\mu\nu}$.
- We have

$$H_{\mu\nu} = \nabla_\tau F_{\mu\rho} \nabla^\tau F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} \nabla_\tau F_{\rho\sigma} \nabla^\tau F^{\rho\sigma} =$$

$$= \frac{1}{2} \left(\nabla_\tau F_{\mu\rho} \nabla^\tau F_\nu{}^\rho + \nabla_\tau \star F_{\mu\rho} \nabla^\tau \star F_\nu{}^\rho \right)$$

- Thus, $H_{\mu\nu} = H_{\nu\mu}$ and $H^{\rho}{}_{\rho} = 0$.
- More importantly, for source-free $F_{\mu\nu}$: $\nabla_\mu H^{\mu\nu} = 0$

Application: new conservation laws for $F_{\mu\nu}$

- Consider the trace of the Chevreton tensor $H_{\mu\nu} = H^{\rho}{}_{\rho\mu\nu}$.
- We have

$$H_{\mu\nu} = \nabla_\tau F_{\mu\rho} \nabla^\tau F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} \nabla_\tau F_{\rho\sigma} \nabla^\tau F^{\rho\sigma} =$$
$$= \frac{1}{2} \left(\nabla_\tau F_{\mu\rho} \nabla^\tau F_\nu{}^\rho + \nabla_\tau \star F_{\mu\rho} \nabla^\tau \star F_\nu{}^\rho \right)$$

- Thus, $H_{\mu\nu} = H_{\nu\mu}$ and $H^{\rho}{}_{\rho} = 0$.
- More importantly, for source-free $F_{\mu\nu}$: $\nabla_\mu H^{\mu\nu} = 0$
- This is valid

Application: new conservation laws for $F_{\mu\nu}$

- Consider the trace of the Chevreton tensor $H_{\mu\nu} = H^{\rho}{}_{\rho\mu\nu}$.
- We have

$$H_{\mu\nu} = \nabla_\tau F_{\mu\rho} \nabla^\tau F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} \nabla_\tau F_{\rho\sigma} \nabla^\tau F^{\rho\sigma} = \\ = \frac{1}{2} \left(\nabla_\tau F_{\mu\rho} \nabla^\tau F_\nu{}^\rho + \nabla_\tau \star F_{\mu\rho} \nabla^\tau \star F_\nu{}^\rho \right)$$

- Thus, $H_{\mu\nu} = H_{\nu\mu}$ and $H^{\rho}{}_{\rho} = 0$.
- More importantly, for source-free $F_{\mu\nu}$: $\nabla_\mu H^{\mu\nu} = 0$
- This is valid
  1. if the electromagnetic field is "test" (spacetime is vacuum with a possible Λ).

Application: new conservation laws for $F_{\mu\nu}$

- Consider the trace of the Chevreton tensor $H_{\mu\nu} = H^{\rho}{}_{\rho\mu\nu}$.
- We have

$$H_{\mu\nu} = \nabla_\tau F_{\mu\rho} \nabla^\tau F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} \nabla_\tau F_{\rho\sigma} \nabla^\tau F^{\rho\sigma} =$$

$$= \frac{1}{2} \left(\nabla_\tau F_{\mu\rho} \nabla^\tau F_\nu{}^\rho + \nabla_\tau \star F_{\mu\rho} \nabla^\tau \star F_\nu{}^\rho \right)$$

- Thus, $H_{\mu\nu} = H_{\nu\mu}$ and $H^{\rho}{}_{\rho} = 0$.
- More importantly, for source-free $F_{\mu\nu}$: $\nabla_\mu H^{\mu\nu} = 0$
- This is valid
  1. if the electromagnetic field is "test" (spacetime is vacuum with a possible Λ).
  2. In the full non-linear Einstein-Maxwell theory with a possible Λ.

Application: new conservation laws for $F_{\mu\nu}$

- Consider the trace of the Chevreton tensor $H_{\mu\nu} = H^{\rho}{}_{\rho\mu\nu}$.
- We have

$$H_{\mu\nu} = \nabla_\tau F_{\mu\rho} \nabla^\tau F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} \nabla_\tau F_{\rho\sigma} \nabla^\tau F^{\rho\sigma} =$$
$$= \frac{1}{2} \left(\nabla_\tau F_{\mu\rho} \nabla^\tau F_\nu{}^\rho + \nabla_\tau \star F_{\mu\rho} \nabla^\tau \star F_\nu{}^\rho \right)$$

- Thus, $H_{\mu\nu} = H_{\nu\mu}$ and $H^\rho{}_\rho = 0$.
- More importantly, for source-free $F_{\mu\nu}$: $\nabla_\mu H^{\mu\nu} = 0$
- This is valid
  1. if the electromagnetic field is "test" (spacetime is vacuum with a possible Λ).
  2. In the full non-linear Einstein-Maxwell theory with a possible Λ.
- Thus, given any conformal Killing vector $\vec{\xi}$

$$J^\mu(\vec{\xi}) \equiv H^{\mu\nu} \xi_\nu \ \Rightarrow \ \nabla_\mu J^\mu = 0$$

$\Longrightarrow$ new conserved quantities in Einstein-Maxwell spacetimes having $\vec{\xi}$.

Further properties of $H_{\mu\nu}$

- One can prove that $H_{\mu\nu} = 0$ if and only if the full Chevreton tensor is of pure radiation type $H_{\alpha\beta\mu\nu} \propto \ell_\alpha \ell_\beta \ell_\mu \ell_\nu$ for null ℓ_μ.

Further properties of $H_{\mu\nu}$

- One can prove that $H_{\mu\nu} = 0$ if and only if the full Chevreton tensor is of pure radiation type $H_{\alpha\beta\mu\nu} \propto \ell_\alpha \ell_\beta \ell_\mu \ell_\nu$ for null ℓ_μ.
- Then, the Petrov type is either N or 0 — in the former case with ℓ_μ the principal null direction and $F_{\mu\nu}$ null—, and Λ must vanish.

Further properties of $H_{\mu\nu}$

- One can prove that $H_{\mu\nu} = 0$ if and only if the full Chevreton tensor is of pure radiation type $H_{\alpha\beta\mu\nu} \propto \ell_\alpha \ell_\beta \ell_\mu \ell_\nu$ for null ℓ_μ.
- Then, the Petrov type is either N or 0 — in the former case with ℓ_μ the principal null direction and $F_{\mu\nu}$ null—, and Λ must vanish.
- For a general null $F_{\mu\nu}$, one has

$$H_{\mu\nu} = \nabla_\rho \left[\ell_{(\mu} \nabla^\rho \ell_{\nu)} - \ell^\rho \nabla_{(\mu} \ell_{\nu)} - \ell_{(\mu} \nabla_{\nu)} \ell^\rho \right] .$$

Further properties of $H_{\mu\nu}$

- One can prove that $H_{\mu\nu} = 0$ if and only if the full Chevreton tensor is of pure radiation type $H_{\alpha\beta\mu\nu} \propto \ell_\alpha \ell_\beta \ell_\mu \ell_\nu$ for null ℓ_μ.
- Then, the Petrov type is either N or 0 — in the former case with ℓ_μ the principal null direction and $F_{\mu\nu}$ null—, and Λ must vanish.
- For a general null $F_{\mu\nu}$, one has

$$H_{\mu\nu} = \nabla_\rho \left[\ell_{(\mu} \nabla^\rho \ell_{\nu)} - \ell^\rho \nabla_{(\mu} \ell_{\nu)} - \ell_{(\mu} \nabla_{\nu)} \ell^\rho \right].$$

- A surprising property is that, in Einstein-Maxwell spacetimes, $H_{\mu\nu}$ is essentially the conformally well-behaved Bach tensor:

$$B_{\mu\nu} = 2H_{\mu\nu} + \frac{2}{3}\Lambda T_{\mu\nu}$$

(recall: $B_{\mu\nu} = \left(\nabla^\rho \nabla^\sigma - \frac{1}{2} R^{\rho\sigma}\right) C_{\mu\rho\nu\sigma}$).

Robinson-Trauman type-D solution with null $F_{\mu\nu}$

- A type-D solution of Einstein-Maxwell eqs. with Λ:

$$ds^2 = r^2(dx^2 + dy^2) - 2dudr + \left(\frac{2m(u)}{r} + \frac{\Lambda}{3}r^2\right)du^2$$

Robinson-Trauman type-D solution with null $F_{\mu\nu}$

- A type-D solution of Einstein-Maxwell eqs. with Λ:

$$ds^2 = r^2(dx^2 + dy^2) - 2dudr + \left(\frac{2m(u)}{r} + \frac{\Lambda}{3}r^2\right) du^2$$

- $m(u)$—the "mass"— is an arbitrary function with $\dot{m} \leq 0$

Robinson-Trauman type-D solution with null $F_{\mu\nu}$

- A type-D solution of Einstein-Maxwell eqs. with Λ:

$$ds^2 = r^2(dx^2 + dy^2) - 2dudr + \left(\frac{2m(u)}{r} + \frac{\Lambda}{3}r^2\right) du^2$$

- $m(u)$—the "mass"— is an arbitrary function with $\dot{m} \leq 0$
- The null electromagnetic field is given by $\boldsymbol{F} = \sqrt{-2\dot{m}}\, du \wedge dx$ and its wave one-form is $\boldsymbol{\ell} = \frac{\sqrt{-2\dot{m}}}{r}\, du$

Robinson-Trauman type-D solution with null $F_{\mu\nu}$

- A type-D solution of Einstein-Maxwell eqs. with Λ:

$$ds^2 = r^2(dx^2 + dy^2) - 2dudr + \left(\frac{2m(u)}{r} + \frac{\Lambda}{3}r^2\right) du^2$$

- $m(u)$—the "mass"— is an arbitrary function with $\dot{m} \leq 0$
- The null electromagnetic field is given by $\boldsymbol{F} = \sqrt{-2\dot{m}}\, du \wedge dx$ and its wave one-form is $\boldsymbol{\ell} = \frac{\sqrt{-2\dot{m}}}{r}\, du$
- Using now the three Killing vectors $\vec{\xi}_i = \{\partial_x, \partial_y, y\partial_x - x\partial_y\}$ one can easily check that all the currents $T_{\mu\nu}\xi_i^\nu$ are identically vanishing

Robinson-Trauman type-D solution with null $F_{\mu\nu}$

- A type-D solution of Einstein-Maxwell eqs. with Λ:

$$ds^2 = r^2(dx^2 + dy^2) - 2dudr + \left(\frac{2m(u)}{r} + \frac{\Lambda}{3}r^2\right) du^2$$

- $m(u)$—the "mass"— is an arbitrary function with $\dot{m} \leq 0$
- The null electromagnetic field is given by $\boldsymbol{F} = \sqrt{-2\dot{m}}\, du \wedge dx$ and its wave one-form is $\boldsymbol{\ell} = \frac{\sqrt{-2\dot{m}}}{r}\, du$
- Using now the three Killing vectors $\vec{\xi}_i = \{\partial_x, \partial_y, y\partial_x - x\partial_y\}$ one can easily check that all the currents $T_{\mu\nu}\xi_i^\nu$ are identically vanishing
- However, the divergence-free currents built with $H_{\mu\nu}$ read

$$J^\mu(\xi_i) = H^\mu{}_\nu \xi_i^\nu = -\frac{2\dot{m}}{r^4}\xi_i^\mu \qquad \text{for all } i = 1, 2, 3$$

and are non-vanishing in general

Robinson-Trauman type-D solution with null $F_{\mu\nu}$

- A type-D solution of Einstein-Maxwell eqs. with Λ:

$$ds^2 = r^2(dx^2 + dy^2) - 2dudr + \left(\frac{2m(u)}{r} + \frac{\Lambda}{3}r^2\right) du^2$$

- $m(u)$—the "mass"— is an arbitrary function with $\dot{m} \leq 0$
- The null electromagnetic field is given by $\boldsymbol{F} = \sqrt{-2\dot{m}}\, du \wedge dx$ and its wave one-form is $\boldsymbol{\ell} = \frac{\sqrt{-2\dot{m}}}{r}\, du$
- Using now the three Killing vectors $\vec{\xi}_i = \{\partial_x, \partial_y, y\partial_x - x\partial_y\}$ one can easily check that all the currents $T_{\mu\nu}\xi_i^\nu$ are identically vanishing
- However, the divergence-free currents built with $H_{\mu\nu}$ read

$$J^\mu(\xi_i) = H^\mu{}_\nu \xi_i^\nu = -\frac{2\dot{m}}{r^4}\xi_i^\mu \qquad \text{for all } i = 1, 2, 3$$

  and are non-vanishing in general
- Thus, there are non-trivial conserved quantities involving the physically relevant magnitude $\dot{m}$ at the super-energy level.

Causal propagation of fields

Let $\mathcal{S}$ be any closed achronal set and $D(\mathcal{S})$ its total Cauchy development. Let $w_\mu = -t_{,\mu}$ be any timelike 1-form foliating $D(\mathcal{S})$ with hypersurfaces $t =$const.

Causal propagation of fields

Let $\mathcal{S}$ be any closed achronal set and $D(\mathcal{S})$ its total Cauchy development. Let $w_\mu = -t_{,\mu}$ be any timelike 1-form foliating $D(\mathcal{S})$ with hypersurfaces t =const.

Theorem (Causal propagation)

If the super-energy tensor $T^{\rho\mu_1\dots\lambda_r\mu_r}\{A\}$ of any tensor field $A_{\mu_1\dots\mu_m}$ satisfies the following divergence condition

$$\nabla_\rho T^{\rho\mu_1\dots\lambda_r\mu_r} w_{\mu_1}\dots w_{\lambda_r} w_{\mu_r} \leq f\, T^{\lambda_1\mu_1\dots\lambda_r\mu_r} w_{\lambda_1} w_{\mu_1}\dots w_{\lambda_r} w_{\mu_r}$$

where f is a continuous function, then

$$A_{\mu_1\dots\mu_m}|_{\mathcal{S}} = 0 \qquad \Longrightarrow \qquad A_{\mu_1\dots\mu_m}|_{\overline{D(\mathcal{S})}} = 0.$$

Causal propagation of fields

Let $\mathcal{S}$ be any closed achronal set and $D(\mathcal{S})$ its total Cauchy development. Let $w_\mu = -t_{,\mu}$ be any timelike 1-form foliating $D(\mathcal{S})$ with hypersurfaces $t =$const.

Theorem (Causal propagation)

If the super-energy tensor $T^{\rho\mu_1\ldots\lambda_r\mu_r}\{A\}$ of any tensor field $A_{\mu_1\ldots\mu_m}$ satisfies the following divergence condition

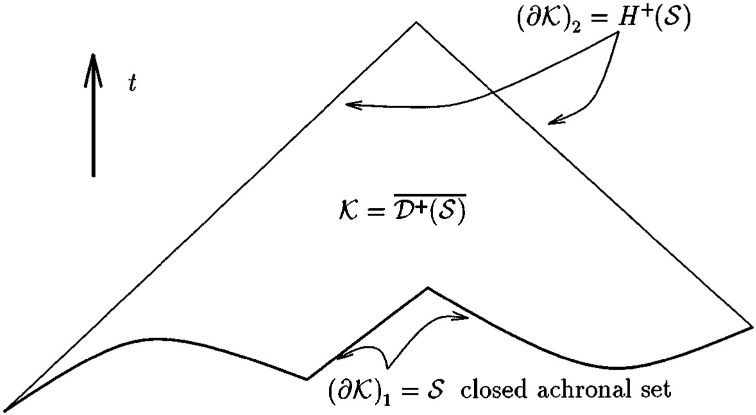
$$\nabla_\rho T^{\rho\mu_1\ldots\lambda_r\mu_r} w_{\mu_1}\ldots w_{\lambda_r} w_{\mu_r} \leq f\, T^{\lambda_1\mu_1\ldots\lambda_r\mu_r} w_{\lambda_1} w_{\mu_1}\ldots w_{\lambda_r} w_{\mu_r}$$

where f is a continuous function, then

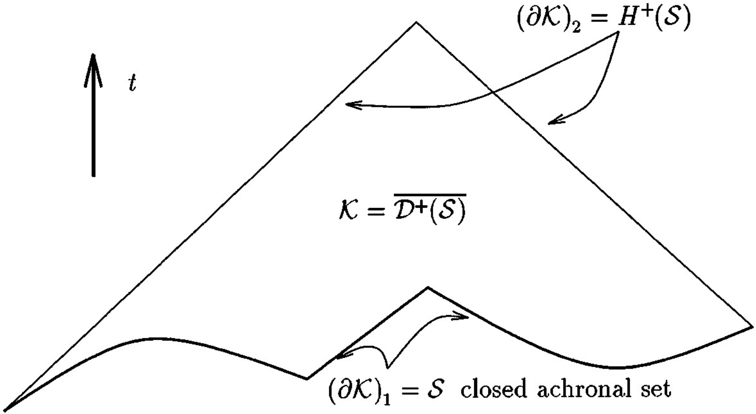
$$A_{\mu_1\ldots\mu_m}|_{\mathcal{S}} = 0 \qquad \Longrightarrow \qquad A_{\mu_1\ldots\mu_m}|_{\overline{D(\mathcal{S})}} = 0.$$

Let us remark that a key point in the proof is the dominant property, which in particular entails that the super-energy-momentum vector $P^\rho = T^{\rho\mu_1\ldots\lambda_r\mu_r} w_{\mu_1}\ldots w_{\lambda_r} w_{\mu_r}$ is future pointing.

Causal propagation of fields



Causal propagation of fields



Observe that, in particular, as the Bel-Robinson tensor is divergence-free in vacuum, one derives the causal propagation of gravity in vacuum (Phys. Rev. Lett. **78** (1997) 783).

What is important about "energy"?

- The properties of energy that make it a fundamental quantity in Physics are:

What is important about "energy"?

- The properties of energy that make it a fundamental quantity in Physics are:
  1. Positivity

What is important about "energy"?

- The properties of energy that make it a fundamental quantity in Physics are:
  1. Positivity
  2. Dominance

What is important about "energy"?

- The properties of energy that make it a fundamental quantity in Physics are:
  1. Positivity
  2. Dominance
  3. "Conservation"

What is important about "energy"?

- The properties of energy that make it a fundamental quantity in Physics are:
  1. Positivity
  2. Dominance
  3. "Conservation"
  4. **Exchange** between fields keeping conservation

What is important about "energy"?

- The properties of energy that make it a fundamental quantity in Physics are:
  1. Positivity
  2. Dominance
  3. "Conservation"
  4. **Exchange** between fields keeping conservation
- Points 1 through 3 are kept by "superenergy" tensors! What about 4?

What is important about "energy"?

- The properties of energy that make it a fundamental quantity in Physics are:
  1. Positivity
  2. Dominance
  3. “Conservation”
  4. **Exchange** between fields keeping conservation
- Points 1 through 3 are kept by "superenergy" tensors! What about 4?
- Recall $\nabla_\alpha B^{\alpha\beta\lambda\mu} = 0$ whenever $J_{\lambda\mu\beta} = 0$

What is important about "energy"?

- The properties of energy that make it a fundamental quantity in Physics are:
  1. Positivity
  2. Dominance
  3. "Conservation"
  4. **Exchange** between fields keeping conservation
- Points 1 through 3 are kept by "superenergy" tensors! What about 4?
- Recall $\nabla_\alpha B^{\alpha\beta\lambda\mu} = 0$ whenever $J_{\lambda\mu\beta} = 0$
- Recall $\nabla_\alpha \mathcal{S}^{\alpha\beta\lambda\mu} = 0$ whenever $R_{\alpha\beta\lambda\mu} = 0$

What is important about "energy"?

- The properties of energy that make it a fundamental quantity in Physics are:
  1. Positivity
  2. Dominance
  3. "Conservation"
  4. **Exchange** between fields keeping conservation
- Points 1 through 3 are kept by "superenergy" tensors! What about 4?
- Recall $\nabla_\alpha B^{\alpha\beta\lambda\mu} = 0$ whenever $J_{\lambda\mu\beta} = 0$
- Recall $\nabla_\alpha \mathcal{S}^{\alpha\beta\lambda\mu} = 0$ whenever $R_{\alpha\beta\lambda\mu} = 0$
- Can one combine the "super-energy" tensor for gravity and physical fields with the aim of restoring conservation if $J_{\lambda\mu\beta} \neq 0$?

What is important about "energy"?

- The properties of energy that make it a fundamental quantity in Physics are:
  1. Positivity
  2. Dominance
  3. “Conservation”
  4. **Exchange** between fields keeping conservation
- Points 1 through 3 are kept by "superenergy" tensors! What about 4?
- Recall $\nabla_\alpha B^{\alpha\beta\lambda\mu} = 0$ whenever $J_{\lambda\mu\beta} = 0$
- Recall $\nabla_\alpha \mathcal{S}^{\alpha\beta\lambda\mu} = 0$ whenever $R_{\alpha\beta\lambda\mu} = 0$
- Can one combine the "super-energy" tensor for gravity and physical fields with the aim of restoring conservation if $J_{\lambda\mu\beta} \neq 0$?
- Again, filling the gap:

	$T_{\mu\nu}$	“superenergy”
Gravity	NO	YES
Physical fields	YES	??

The Einstein-Klein-Gordon case

- Consider a minimally coupled scalar field ϕ with mass m (m can be zero), so that the Einstein field equations hold:

$$R_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{n-2} m^2 \phi^2 g_{\mu\nu}$$

The Einstein-Klein-Gordon case

- Consider a minimally coupled scalar field ϕ with mass m (m can be zero), so that the Einstein field equations hold:

$$R_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{n-2} m^2 \phi^2 g_{\mu\nu}$$

- This implies the Klein-Gordon equation $\nabla^\rho \nabla_\rho \phi = m^2 \phi$.

The Einstein-Klein-Gordon case

- Consider a minimally coupled scalar field ϕ with mass m (m can be zero), so that the Einstein field equations hold:

$$R_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{n-2} m^2 \phi^2 g_{\mu\nu}$$

- This implies the Klein-Gordon equation $\nabla^\rho \nabla_\rho \phi = m^2 \phi$.
- Then, the matter current is

$$J_{\lambda\mu\beta} = 2 \nabla_\beta \nabla_{[\lambda} \phi \, \nabla_{\mu]} \phi + \frac{4}{n-2} m^2 \phi \, g_{\beta[\mu} \nabla_{\lambda]} \phi$$

The Einstein-Klein-Gordon case

- Consider a minimally coupled scalar field ϕ with mass m (m can be zero), so that the Einstein field equations hold:

$$R_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{n-2} m^2 \phi^2 g_{\mu\nu}$$

- This implies the Klein-Gordon equation $\nabla^\rho \nabla_\rho \phi = m^2 \phi$.
- Then, the matter current is

$$J_{\lambda\mu\beta} = 2\nabla_\beta \nabla_{[\lambda}\phi \, \nabla_{\mu]}\phi + \frac{4}{n-2} m^2 \phi \, g_{\beta[\mu} \nabla_{\lambda]}\phi$$

- Thus, the divergence of the Bel tensor becomes in this case

$$\begin{aligned}
\nabla_\alpha B^\alpha{}_{\beta\lambda\mu} = {} & 2\nabla_\sigma \phi \nabla^\rho \nabla_{(\lambda} \phi R^\sigma{}_{\mu)\rho\beta} \\
& - g_{\lambda\mu} \nabla_\sigma \phi \nabla^\rho \nabla^\tau \phi R^\sigma{}_{\tau\rho\beta} + 2\nabla^\sigma \nabla^\rho \phi R_{\beta\rho\sigma(\lambda} \nabla_{\mu)} \phi \\
& - \frac{2}{n-2} m^2 \phi \Big[2\nabla_\sigma \phi R^\sigma{}_{(\lambda\mu)\beta} - 2\nabla_\beta \phi \nabla_\lambda \phi \nabla_\mu \phi - \\
& - \frac{2}{n-2} m^2 \phi^2 g_{\beta(\lambda} \nabla_{\mu)} \phi + g_{\lambda\mu} \nabla_\beta \phi \left(\nabla_\rho \phi \nabla^\rho \phi + \frac{1}{n-2} m^2 \phi^2 \right) \Big].
\end{aligned}$$

The Einstein-Klein-Gordon case

- The super-energy tensor of the scalar field is given by

$$\mathcal{S}_{\alpha\beta\lambda\mu} = 2\nabla_\alpha\nabla_{(\lambda}\phi\nabla_{\mu)}\nabla_\beta\phi - g_{\alpha\beta}\left(\nabla_\lambda\nabla^\rho\phi\nabla_\mu\nabla_\rho\phi + m^2\nabla_\lambda\phi\nabla_\mu\phi\right)$$
$$-g_{\lambda\mu}\left(\nabla_\alpha\nabla^\rho\phi\nabla_\beta\nabla_\rho\phi + m^2\nabla_\alpha\phi\nabla_\beta\phi\right)$$
$$+\frac{1}{2}g_{\alpha\beta}g_{\lambda\mu}\left(\nabla_\sigma\nabla_\rho\phi\nabla^\sigma\nabla^\rho\phi + 2m^2\nabla_\rho\phi\nabla^\rho\phi + m^4\phi^2\right)$$

The Einstein-Klein-Gordon case

- The super-energy tensor of the scalar field is given by

$$\mathcal{S}_{\alpha\beta\lambda\mu} = 2\nabla_\alpha\nabla_{(\lambda}\phi\nabla_{\mu)}\nabla_\beta\phi - g_{\alpha\beta}\left(\nabla_\lambda\nabla^\rho\phi\nabla_\mu\nabla_\rho\phi + m^2\nabla_\lambda\phi\nabla_\mu\phi\right)$$
$$-g_{\lambda\mu}\left(\nabla_\alpha\nabla^\rho\phi\nabla_\beta\nabla_\rho\phi + m^2\nabla_\alpha\phi\nabla_\beta\phi\right)$$
$$+\frac{1}{2}g_{\alpha\beta}g_{\lambda\mu}\left(\nabla_\sigma\nabla_\rho\phi\nabla^\sigma\nabla^\rho\phi + 2m^2\nabla_\rho\phi\nabla^\rho\phi + m^4\phi^2\right)$$

- Its divergence reads

$$\nabla_\alpha\mathcal{S}^\alpha{}_{\beta\lambda\mu} = -2\nabla_\sigma\phi\nabla^\rho\nabla_{(\lambda}\phi R^\sigma{}_{\mu)\rho\beta} + g_{\lambda\mu}\nabla_\sigma\phi\nabla^\rho\nabla^\tau\phi R^\sigma{}_{\tau\rho\beta} +$$
$$+\left(\nabla_\rho\phi\nabla^\rho\phi + \frac{1}{n-2}m^2\phi^2\right)\left(2\nabla_\beta\nabla_{(\lambda}\phi\,\nabla_{\mu)}\phi - g_{\lambda\mu}\nabla_\beta\nabla_\rho\phi\,\nabla^\rho\phi\right).$$

The Einstein-Klein-Gordon case

- The super-energy tensor of the scalar field is given by

$$\mathcal{S}_{\alpha\beta\lambda\mu} = 2\nabla_\alpha\nabla_{(\lambda}\phi\nabla_{\mu)}\nabla_\beta\phi - g_{\alpha\beta}\left(\nabla_\lambda\nabla^\rho\phi\nabla_\mu\nabla_\rho\phi + m^2\nabla_\lambda\phi\nabla_\mu\phi\right)$$
$$-g_{\lambda\mu}\left(\nabla_\alpha\nabla^\rho\phi\nabla_\beta\nabla_\rho\phi + m^2\nabla_\alpha\phi\nabla_\beta\phi\right)$$
$$+\frac{1}{2}g_{\alpha\beta}g_{\lambda\mu}\left(\nabla_\sigma\nabla_\rho\phi\nabla^\sigma\nabla^\rho\phi + 2m^2\nabla_\rho\phi\nabla^\rho\phi + m^4\phi^2\right)$$

- Its divergence reads

$$\nabla_\alpha\mathcal{S}^\alpha{}_{\beta\lambda\mu} = -2\nabla_\sigma\phi\nabla^\rho\nabla_{(\lambda}\phi R^\sigma{}_{\mu)\rho\beta} + g_{\lambda\mu}\nabla_\sigma\phi\nabla^\rho\nabla^\tau\phi R^\sigma{}_{\tau\rho\beta} +$$
$$+\left(\nabla_\rho\phi\nabla^\rho\phi + \frac{1}{n-2}m^2\phi^2\right)\left(2\nabla_\beta\nabla_{(\lambda}\phi\,\nabla_{\mu)}\phi - g_{\lambda\mu}\nabla_\beta\nabla_\rho\phi\,\nabla^\rho\phi\right).$$

- One can then check that the direct sum $B_{\alpha\beta\lambda\mu} + \mathcal{S}_{\alpha\beta\lambda\mu}$ is *not* divergence-free in general.

The Einstein-Klein-Gordon case

- The super-energy tensor of the scalar field is given by

$$\mathcal{S}_{\alpha\beta\lambda\mu} = 2\nabla_\alpha\nabla_{(\lambda}\phi\nabla_{\mu)}\nabla_\beta\phi - g_{\alpha\beta}\left(\nabla_\lambda\nabla^\rho\phi\nabla_\mu\nabla_\rho\phi + m^2\nabla_\lambda\phi\nabla_\mu\phi\right)$$
$$-g_{\lambda\mu}\left(\nabla_\alpha\nabla^\rho\phi\nabla_\beta\nabla_\rho\phi + m^2\nabla_\alpha\phi\nabla_\beta\phi\right)$$
$$+\frac{1}{2}g_{\alpha\beta}g_{\lambda\mu}\left(\nabla_\sigma\nabla_\rho\phi\nabla^\sigma\nabla^\rho\phi + 2m^2\nabla_\rho\phi\nabla^\rho\phi + m^4\phi^2\right)$$

- Its divergence reads

$$\nabla_\alpha\mathcal{S}^\alpha{}_{\beta\lambda\mu} = -2\nabla_\sigma\phi\nabla^\rho\nabla_{(\lambda}\phi R^\sigma{}_{\mu)\rho\beta} + g_{\lambda\mu}\nabla_\sigma\phi\nabla^\rho\nabla^\tau\phi R^\sigma{}_{\tau\rho\beta} +$$
$$+\left(\nabla_\rho\phi\nabla^\rho\phi + \frac{1}{n-2}m^2\phi^2\right)\left(2\nabla_\beta\nabla_{(\lambda}\phi\,\nabla_{\mu)}\phi - g_{\lambda\mu}\nabla_\beta\nabla_\rho\phi\,\nabla^\rho\phi\right).$$

- One can then check that the direct sum $B_{\alpha\beta\lambda\mu} + \mathcal{S}_{\alpha\beta\lambda\mu}$ is *not* divergence-free in general.
- However, this is not relevant. Conservation arises if there are symmetries!

The Einstein-Klein-Gordon case

- Assume $\vec{\xi}$ is a Killing vector. Then it is known that

$$\xi^\mu \nabla_\mu \phi = 0,$$

The Einstein-Klein-Gordon case

- Assume $\vec{\xi}$ is a Killing vector. Then it is known that

$$\xi^{\mu}\nabla_{\mu}\phi = 0,$$

- it also follows that

$$\xi^{\beta}\nabla^{\rho}\phi\,\nabla_{\beta}\nabla_{\rho}\phi = 0$$

The Einstein-Klein-Gordon case

- Assume $\vec{\xi}$ is a Killing vector. Then it is known that

$$\xi^{\mu}\nabla_{\mu}\phi = 0,$$

- it also follows that

$$\xi^{\beta}\nabla^{\rho}\phi\,\nabla_{\beta}\nabla_{\rho}\phi = 0$$

The Einstein-Klein-Gordon case

- Assume $\vec{\xi}$ is a Killing vector. Then it is known that

$$\xi^\mu \nabla_\mu \phi = 0,$$

- it also follows that

$$\xi^\beta \nabla^\rho \phi \, \nabla_\beta \nabla_\rho \phi = 0$$

Then

$$\xi^\beta \xi^\lambda \xi^\mu \nabla_\alpha B^\alpha{}_{\beta\lambda\mu} = \nabla_\sigma \phi \left(2 \nabla_\rho \nabla_\lambda \phi R^\sigma{}_\mu{}^\rho{}_\beta + g_{\lambda\mu} R^{\sigma\rho}{}_\beta{}^\tau \nabla_\rho \nabla_\tau \phi\right) \xi^\beta \xi^\lambda \xi^\mu,$$
$$\xi^\beta \xi^\lambda \xi^\mu \nabla_\alpha \mathcal{S}^\alpha{}_{\beta\lambda\mu} = -\nabla_\sigma \phi \left(2 \nabla_\rho \nabla_\lambda \phi R^\sigma{}_\mu{}^\rho{}_\beta + g_{\lambda\mu} R^{\sigma\rho}{}_\beta{}^\tau \nabla_\rho \nabla_\tau \phi\right) \xi^\beta \xi^\lambda \xi^\mu$$

The Einstein-Klein-Gordon case

- Assume $\vec{\xi}$ is a Killing vector. Then it is known that

$$\xi^{\mu}\nabla_{\mu}\phi = 0,$$

- it also follows that

$$\xi^{\beta}\nabla^{\rho}\phi\,\nabla_{\beta}\nabla_{\rho}\phi = 0$$

Then

$$\xi^{\beta}\xi^{\lambda}\xi^{\mu}\nabla_{\alpha}B^{\alpha}{}_{\beta\lambda\mu}=\nabla_{\sigma}\phi\,(2\nabla_{\rho}\nabla_{\lambda}\phi R^{\sigma}{}_{\mu}{}^{\rho}{}_{\beta}+g_{\lambda\mu}R^{\sigma\rho}{}_{\beta}{}^{\tau}\nabla_{\rho}\nabla_{\tau}\phi)\,\xi^{\beta}\xi^{\lambda}\xi^{\mu},$$
$$\xi^{\beta}\xi^{\lambda}\xi^{\mu}\nabla_{\alpha}\mathcal{S}^{\alpha}{}_{\beta\lambda\mu}=-\nabla_{\sigma}\phi\,(2\nabla_{\rho}\nabla_{\lambda}\phi R^{\sigma}{}_{\mu}{}^{\rho}{}_{\beta}+g_{\lambda\mu}R^{\sigma\rho}{}_{\beta}{}^{\tau}\nabla_{\rho}\nabla_{\tau}\phi)\,\xi^{\beta}\xi^{\lambda}\xi^{\mu}$$

Hence:

$$\xi^{\beta}\xi^{\lambda}\xi^{\mu}\nabla_{\alpha}\left(B^{\alpha}{}_{\beta\lambda\mu}+\mathcal{S}^{\alpha}{}_{\beta\lambda\mu}\right)=0.$$

A mixed conserved current!

- Using the symmetry properties of the super-energy tensors

$$\nabla_\alpha \left[\left(B^\alpha{}_{\beta\lambda\mu} + \mathcal{S}^\alpha{}_{\beta\lambda\mu} \right) \xi^\beta \xi^\lambda \xi^\mu \right] = 0$$

A mixed conserved current!

- Using the symmetry properties of the super-energy tensors

$$\nabla_\alpha \left[\left(B^\alpha{}_{\beta\lambda\mu} + \mathcal{S}^\alpha{}_{\beta\lambda\mu} \right) \xi^\beta \xi^\lambda \xi^\mu \right] = 0$$

- Observe that only the completely symmetric part of the super-energy tensors is relevant here

A mixed conserved current!

- Using the symmetry properties of the super-energy tensors

$$\nabla_\alpha \left[\left(B^\alpha{}_{\beta\lambda\mu} + \mathcal{S}^\alpha{}_{\beta\lambda\mu} \right) \xi^\beta \xi^\lambda \xi^\mu \right] = 0$$

- Observe that only the completely symmetric part of the super-energy tensors is relevant here
- Therefore, the super-energy currents

$$j_\alpha \equiv \left(B_{\alpha\beta\lambda\mu} + \mathcal{S}_{\alpha\beta\lambda\mu} \right) \xi^\beta \xi^\lambda \xi^\mu$$

are divergence-free.

A mixed conserved current!

- Using the symmetry properties of the super-energy tensors

$$\nabla_\alpha \left[\left(B^\alpha{}_{\beta\lambda\mu} + \mathcal{S}^\alpha{}_{\beta\lambda\mu} \right) \xi^\beta \xi^\lambda \xi^\mu \right] = 0$$

- Observe that only the completely symmetric part of the super-energy tensors is relevant here
- Therefore, the super-energy currents

$$j_\alpha \equiv \left(B_{\alpha\beta\lambda\mu} + \mathcal{S}_{\alpha\beta\lambda\mu} \right) \xi^\beta \xi^\lambda \xi^\mu$$

are divergence-free.
- This leads to conservation via exchange of super-energy!

A mixed conserved current!

- Using the symmetry properties of the super-energy tensors

$$\nabla_\alpha \left[\left(B^\alpha{}_{\beta\lambda\mu} + \mathcal{S}^\alpha{}_{\beta\lambda\mu} \right) \xi^\beta \xi^\lambda \xi^\mu \right] = 0$$

- Observe that only the completely symmetric part of the super-energy tensors is relevant here
- Therefore, the super-energy currents

$$j_\alpha \equiv \left(B_{\alpha\beta\lambda\mu} + \mathcal{S}_{\alpha\beta\lambda\mu} \right) \xi^\beta \xi^\lambda \xi^\mu$$

  are divergence-free.
- This leads to conservation via exchange of super-energy!
- Actually, one can actually use any three Killing vectors (if they are available) and the currents

$$j_\alpha \equiv \left(B_{(\alpha\beta\lambda\mu)} + \mathcal{S}_{(\alpha\beta\lambda\mu)} \right) \xi_1^\beta \xi_2^\lambda \xi_3^\mu$$

  are divergence-free in general.

Conclusions and comments

- There is a basically *unique* construction (valid for any seed tensor, in arbitrary dimensions, independent of field equations) providing super-energy tensors with the dominant property.

Conclusions and comments

- There is a basically *unique* construction (valid for any seed tensor, in arbitrary dimensions, independent of field equations) providing super-energy tensors with the dominant property.
- There are many applications: a recent one is the construction of **quality factors** measuring the departure of any stationary metric from the Kerr metric. (GRG **45** (2013) 1095-1127)

Conclusions and comments

- There is a basically *unique* construction (valid for any seed tensor, in arbitrary dimensions, independent of field equations) providing super-energy tensors with the dominant property.
- There are many applications: a recent one is the construction of **quality factors** measuring the departure of any stationary metric from the Kerr metric. (GRG **45** (2013) 1095-1127)
- The correct physical dimensions for any given super-energy density is energy density times L^{-2}.

Conclusions and comments

- There is a basically *unique* construction (valid for any seed tensor, in arbitrary dimensions, independent of field equations) providing super-energy tensors with the dominant property.
- There are many applications: a recent one is the construction of **quality factors** measuring the departure of any stationary metric from the Kerr metric. (GRG **45** (2013) 1095-1127)
- The correct physical dimensions for any given super-energy density is energy density times L^{-2}.
- The most important point is that super-energy tensors give rise to divergence-free currents if the field generating them is isolated while these currents can be *combined* to produce divergence-free currents mixing different fields in interaction.

Conclusions and comments

- There is a basically *unique* construction (valid for any seed tensor, in arbitrary dimensions, independent of field equations) providing super-energy tensors with the dominant property.
- There are many applications: a recent one is the construction of **quality factors** measuring the departure of any stationary metric from the Kerr metric. (GRG **45** (2013) 1095-1127)
- The correct physical dimensions for any given super-energy density is energy density times L^{-2}.
- The most important point is that super-energy tensors give rise to divergence-free currents if the field generating them is isolated while these currents can be *combined* to produce divergence-free currents mixing different fields in interaction.
- Hence, the interchange of super-energy quantities (in a wide sense: they can be super-momentum, or super-stresses etc.) does happen, and the super-energy features can be transferred from one field to another, such as energy properties do.

¡Gracias!

Thank you for your attention

dziękuję !